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Finite Abelian actions on surfaces

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Abstract

Edmonds showed that two free orientation preserving smooth actions φ_1 and φ_2 of a finite Abelian group G on a closed connected oriented smooth surface M are equivalent by an equivariant orientation preserving diffeomorphism iff they have the same bordism class $[M, \varphi_1] = [M, \varphi_2]$ in the oriented bordism group $\Omega_2(G)$ of the group G. In this paper, we compute the bordism class $[M, \varphi]$ for any such action of G on M and we determine for a given M, the bordism classes in $\Omega_2(G)$ that are representable by such actions of G on M. This will enable us to obtain a formula for the number of inequivalent such actions of G on M. We also determine the "weak" equivalence classes of such actions of G on M when all the p-Sylow subgroups of G are homocyclic (i.e. of the form $(\mathbf{Z}/p^\alpha\mathbf{Z})^n$).

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Let M be a closed connected smooth oriented manifold and let $H^+(M)$ (respectively $\mathrm{Diff}^+(M)$) be the group of orientation preserving homeomorphisms (respectively diffeomorphisms) of M and let G be a finite group and set

$$\mathrm{FA}^+(G, M) = \{\varphi \in \mathrm{Hom}(G, H^+(M)) \colon \varphi \text{ injective and } \varphi(g) \text{ is fixed point free for all } 1 \neq g \in G\}$$

$$\text{(respectively } \mathrm{FA}^+_{\mathrm{Diff}}(G, M) = \mathrm{Hom}(G, \mathrm{Diff}^+(M)) \cap \mathrm{FA}^+(G, M)).$$

Note that $\mathrm{Aut}(G)$ (= the automorphism group of G) and $H^+(M)$ (respectively $\mathrm{Diff}^+(M)$) act on $\mathrm{FA}^+(G, M)$ (respectively $\mathrm{FA}^+_{\mathrm{Diff}}(G, M)$) by

$$\varphi.a = \varphi \circ a \quad \text{and} \quad h.\varphi(g) = h \circ \varphi(g) \circ h^{-1} \quad \text{for all } g \in G$$

where $a \in \mathrm{Aut}(G)$, $h \in H^+(M)$ (respectively $\mathrm{Diff}^+(M)$), $\varphi \in \mathrm{FA}^+(G, M)$ (respectively $\mathrm{FA}^+_{\mathrm{Diff}}(G, M)$) and that these two actions commute [2, p. 55]. The set $H^+(M)\backslash \mathrm{FA}^+(G, M)$ (respectively $\mathrm{Diff}^+(M)\backslash \mathrm{FA}^+_{\mathrm{Diff}}(G, M)$) is the set of equivalence classes of free orientation preserving (respectively smooth) actions of G on M and the set

$$H^+(M)\backslash \mathrm{FA}^+(G, M)/\mathrm{Aut}(G) \quad \text{(respectively } \mathrm{Diff}^+(M)\backslash \mathrm{FA}^+_{\mathrm{Diff}}(G, M)/\mathrm{Aut}(G))$$

is the set of weak equivalence classes of free orientation preserving (respectively smooth) actions of G on M.

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A complete classification of $\mathrm{FA}^+(G, M)$ (respectively $\mathrm{FA}^+_{\mathrm{Diff}}(G, M)$) consists therefore of solving the following three problems:

Problem 1. Find necessary and sufficient conditions for $\mathrm{FA}^+(G, M)$ (respectively $\mathrm{FA}^+_{\mathrm{Diff}}(G, M)) \neq \emptyset$.

Problem 2. Determine the set $H^+(M)\backslash \mathrm{FA}^+(G, M)$ (respectively $\mathrm{Diff}^+(M)\backslash \mathrm{FA}^+_{\mathrm{Diff}}(G, M)$).

Problem 3. Determine the set $H^+(M)\backslash \mathrm{FA}^+(G, M)/\operatorname{Aut}(G)$ (respectively $\mathrm{Diff}^+(M)\backslash \mathrm{FA}^+_{\mathrm{Diff}}(G, M)/\operatorname{Aut}(G)$).

The classification problems of $\mathrm{FA}^+(G, M)$ and $\mathrm{FA}^+_{\mathrm{Diff}}(G, M)$ are identical for dimension $M \leqslant 3$ by virtue of the following result.

Proposition 0. *Let G be a finite group and let M be a closed, connected, oriented, smooth manifold of dimension $\leqslant 3$, then the natural inclusion $\mathrm{FA}^+_{\mathrm{Diff}}(G, M) \to \mathrm{FA}^+(G, M)$ induces bijections*

$$\mathrm{Diff}^+(M)\backslash \mathrm{FA}^+_{\mathrm{Diff}}(G, M) \to H^+(M)\backslash \mathrm{FA}^+(G, M)$$

and

$$\mathrm{Diff}^+(M)\backslash \mathrm{FA}^+_{\mathrm{Diff}}(G, M)/\operatorname{Aut}(G) \to H^+(M)\backslash \mathrm{FA}^+(G, M)/\operatorname{Aut}(G).$$

Proof. Clearly it suffices to prove the first bijection.

Surjectivity follows from the uniqueness of the smooth structure on these manifolds (for dimension 1 [8, p. 11]; for dimension 2 [20, pp. 37 and 60] and [25]; for dimension 3 [20, pp. 165 and 252] and [25]). Injectivity follows from the theory of covering spaces by virtue of the density of $\mathrm{Diff}^+(M)$ in $H^+(M)$ on such manifolds [25, Corollary 1, 1.18]. □

In this paper, we consider the classification problem of $\mathrm{FA}^+(G, M)$ for G a finite Abelian group and M a closed, connected oriented surface. Problem 1, is easily solved in Theorem 1.5 and Problem 2, is solved in Theorem 3.4. We give a solution for Problem 3 in Theorem 3.6 when all the p-Sylow subgroups of G are homocyclic (i.e. of the form $(\mathbf{Z}/p^\alpha\mathbf{Z})^n$). Edmonds [11] showed that the bordism map $B : \mathrm{FA}^+_{\mathrm{Diff}}(G, M) \to \Omega_2(G)$ defined by $B(\varphi) = [M, \varphi]$ induces an injection of $\mathrm{Diff}^+(M)\backslash \mathrm{FA}^+_{\mathrm{Diff}}(G, M)$ into the oriented bordism group $\Omega_2(G)$ of G. We compute this bordism map in Theorem 1.7 and we determine its image in Theorem 3.4 where we also obtain a formula for $|H^+(M)\backslash \mathrm{FA}^+(G, M)|$. Problem 2 was solved earlier only when G is cyclic [22] or $G = (\mathbf{Z}/p\mathbf{Z})^n$ [16, Theorem 2.5], [7, Theorem 9] and [23] or in the "low genus case" in [27]. Problem 3 was solved in [17, p. 502] and [7, Corollary 12] when all the p-Sylow subgroups of G are elementary Abelian p-groups (the result claimed in [15, Proposition 4.6] is incorrect, so is the result stated in [11, Remark 4.5]).

This paper is divided into three sections. In Section 1 we introduce the bordism invariant of finite group actions on a closed connected oriented surface and we compute this invariant for the case of a finite Abelian group acting freely on the surface in Theorem 1.7. In Section 2 we study symplectic geometry over a local ring and we obtain our main theorem, Theorem 2.7, which will enable us for any commutative ring A and any maximal ideal m of A and any free A-module V of rank $2g$ and for $G = A/m^{\alpha_1} \oplus \cdots \oplus A/m^{\alpha_k}$ where $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_k > 0$ to determine the set $\mathrm{EHom}_A(V, G)/\mathrm{Sp}_{2g}(A)$ where $\mathrm{EHom}_A(V, G)$ is the set of A-module homomorphisms of V onto G and where

$$\mathrm{Sp}_{2g}(A) = \{L \in M_{2g}(A) : {}^tLJL = J\} \quad \text{with } J = \left[\begin{array}{c|c} & I_g \\ \hline -I_g & \end{array}\right]$$

and where $\mathrm{Sp}_{2g}(A)$ acts to the right on $\mathrm{EHom}_A(V, G)$ as follows:

Let $E = \{e_1, \ldots, e_{2g}\}$ be a basis of V, then $\varphi.L = \varphi \circ S_L$ where $\varphi \in \mathrm{EHom}_A(V, G)$ and $L \in \mathrm{Sp}_{2g}(A)$ and S_L is the A-module automorphism of V whose matrix with respect to the basis E [3, p. 148] is L. Solution of Problem 2 then follows easily from this.

In Section 3 we give the solution of Problem 2 in Theorem 3.4 and of Problem 3 when all the p-Sylow subgroups of G are homocyclic (i.e. of the form $(\mathbf{Z}/p^\alpha\mathbf{Z})^n$) in Theorem 3.6.

1. The bordism invariant

First we remark that the classification problem for the sphere S^2 is trivial.

Proposition 1.1. *Let G be any group, then* $\mathrm{FA}^+(G, S^2) \neq \emptyset$ *iff G is the trivial group.*

Proof. We need only to observe that if $h \in H^+(S^2)$ then the Lefschetz number of h, $\lambda(h) = 2$, hence h must fix some point of S^2 by Lefschetz fixed point theorem [24, p. 195]. □

Next we fix a model for the closed connected orientable surface of genus $g \geqslant 1$. Fix $g \geqslant 1$ and let P_g be the convex hull of $\{x_k = \exp(\frac{2\pi i k}{4g}) \in \mathbf{C}\colon\ 0 \leqslant k \leqslant 4g\}$ in $\mathbf{C}$. Since P_g is convex, if $y_0, \ldots, y_q \in P_g$ we may define $(y_0, \ldots, y_q)$, the linear singular q-simplex of P_g, $(y_0, \ldots, y_q) : \Delta^q \to P_g$ by $(y_0, \ldots, y_q)\ (b_i) = y_i$ for $0 \leqslant i \leqslant q$ [24, p. 115] where $\Delta^q = |b_0, \ldots, b_q|$. Let R_g be the equivalence relation on P_g generated by

$$\bigcup_{i=0}^{g-1} \bigl\{\bigl((x_{4i+1}, x_{4i})(t), (x_{4i+2}, x_{4i+3})(t)\bigr)\colon\ t \in \Delta^1\bigr\} \cup \bigl\{\bigl((x_{4i+2}, x_{4i+1})(t), (x_{4i+3}, x_{4i+4})(t)\bigr)\colon\ t \in \Delta^1\bigr\}$$

and we let $X_g = P_g/R_g$. In particular, $\{x_i\colon\ \{0 \leqslant i \leqslant 4g\} \subseteq P_g$ is identified to a single point in X_g which we denote by $*$. $(X_g, *)$ is the model for the closed connected orientable surface of genus $g \geqslant 1$ with base point $*$ which we fix throughout this paper. By abuse of notation, if $y_0, \ldots, y_q \in P_g$ and if $\pi : P_g \to X_g$ is the canonical map we shall denote by $(y_0, \ldots, y_q)$ the singular q-simplex of X_g defined by $\pi(y_0, \ldots, y_q)$. For $0 \leqslant k < g$ let a_{k+1} (respectively b_{k+1}) be the pointed homotopy class of the loop determined by (x_{4k+2}, x_{4k+3}) (respectively (x_{4k+3}, x_{4k+4})) in $\pi_1(X_g, *)$. An application of Van Kampen theorem gives

$$\pi_1(X_g, *) = \left\langle a_1, b_1, \ldots, a_g, b_g\colon\ \prod_{i=1}^{g} [a_i, b_i] = 1 \right\rangle.$$

For $0 \leqslant k < g$ and by abuse of notation we also let a_{k+1} (respectively b_{k+1}) be the homology class determined by the cycles (x_{4k+2}, x_{4k+3}) (respectively (x_{4k+3}, x_{4k+4})) in $H_1(X_g)$. An application of Mayer–Vietoris sequence shows that $\{a_i, b_i\colon\ 1 \leqslant i \leqslant g\}$ is a basis of $H_1(X_g)$ which we fix throughout this paper.

Define the 2-chain σ in X_g by

$$\sigma = \sum_{i=0}^{g-1} \sigma_i + \sum_{i=0}^{g-3} (x_0, x_{4i+4}, x_{4i+8})$$

where

$$\sigma_i = (x_{4i+2}, x_{4i+3}, x_{4i+4}) - (x_{4i+2}, x_{4i+1}, x_{4i}) - (x_{4i+2}, x_{4i}, x_{4i+4})$$

for $0 \leqslant i < g$ and where the second term drops out if $g \leqslant 2$. Note that $\partial\sigma_i = -(x_{4i}, x_{4i+4})$ for $0 \leqslant i < g$ so that σ is a 2-cycle in X_g. Since $H^i(X_g) = \mathrm{Hom}(H_i(X_g), \mathbf{Z})$ for all i by [24, p. 243] we may choose $\{a_i', b_i'\colon\ 1 \leqslant i \leqslant g\} \subseteq H^1(X_g)$ a basis dual to $\{a_i, b_i\colon\ 1 \leqslant i \leqslant g\} \subseteq H_1(X_g)$. Using the Alexander–Whitney diagonal approximation [24, p. 250] we get $(a_i' \cup b_i')([\sigma]) = 1$ for $1 \leqslant i \leqslant g$, so that $z_g = [\sigma] \in H_2(X_g)$ is a fundamental class of X_g which we fix throughout this paper.

Let $h \in H(X_g)$, the homeomorphism group of X_g, and let w be a path in X_g such that $w(0) = *$ and $w(1) = h(*)$, then $h_{[w]} \circ h_\# \in \mathrm{Aut}(\pi_1(X_g, *))$ where $h_{[w]}$ is the canonical homomorphism defined in [24, p. 382]. Dehn–Nielson theorem [26, p. 194] shows that for $g \geqslant 1$, the function

$$H(X_g)/H_0(X_g) \to \mathrm{Aut}\bigl(\pi_1(X_g, *)\bigr)/\mathrm{IA}\bigl(\pi_1(X_g, *)\bigr)$$

defined by $[h] \to [h_{[w]} \circ h_\#]$ is a group isomorphism, where $H_0(X_g)$ is the group of all homeomorphisms of X_g that are homotopic to the identity and where $\mathrm{IA}(\pi_1(X_g, *))$ is the group of inner automorphisms of $\pi_1(X_g, *)$. This last isomorphism defines an isomorphism

$$H^+(X_g)/H_0(X_g) \to \mathrm{Aut}^+\bigl(\pi_1(X_g, *)\bigr)/\mathrm{IA}\bigl(\pi_1(X_g, *)\bigr)$$

where $\mathrm{Aut}^+(\pi_1(X_g, *))$ is defined by this isomorphism.

Let $\mathrm{EHom}(\pi_1(X_g,*),G)$ be the set of all homomorphisms of $\pi_1(X_g,*)$ onto G, then $\mathrm{Aut}(G)$ (respectively $\mathrm{Aut}^+(\pi_1(X_g,*))$) acts to the left (respectively right) on $\mathrm{EHom}(\pi_1(X_g,*),G)$ by $a.\psi = a\circ\psi$ (respectively $\psi.\varphi = \psi\circ\varphi$) for $a\in\mathrm{Aut}(G)$, $\psi\in\mathrm{EHom}(\pi_1(X_g,*),G)$ and $\varphi\in\mathrm{Aut}^+(\pi_1(X_g,*))$ and these two actions commute.

The following two propositions provide the basis for solving our classification problem.

Proposition 1.2. *Let G be a finite group and let $\hat g\geqslant 1$ and let $\hat g-1=|G|(g-1)$. There is a one-to-one correspondence between the set*

$$H^+(X_{\tilde g})\backslash \mathrm{FA}^+(G,X_{\tilde g})\quad (\textit{respectively } H^+(X_{\tilde g})\backslash \mathrm{FA}^+(G,X_{\tilde g})/\mathrm{Aut}(G))$$

and the set

$$\mathrm{EHom}\big(\pi_1(X_g,*),G\big)/\mathrm{Aut}^+\big(\pi_1(X_g,*)\big)\quad (\textit{respectively } \mathrm{Aut}(G)\backslash \mathrm{EHom}\big(\pi_1(X_g,*),G\big)/\mathrm{Aut}^+\big(\pi_1(X_g,*)\big)).$$

Proof. If $\varphi\in\mathrm{FA}^+(G,X_{\tilde g})$, then the orbit space $X_{\tilde g}/\varphi(G)$ is a closed connected orientable surface of genus g which is given by the Riemann–Hurwitz formula $\hat g-1=|G|(g-1)$. Choose a homeomorphism $\theta: X_{\tilde g}/\varphi(G)\to X_g$ such that we get an orientation preserving covering projection $p:(X_{\tilde g},\tilde *)\to(X_g,*)$ so that $p_*(z_{\hat g})=|G|z_g$ [12, p. 372] and with $\varphi(G)$ as the group of covering transformations. We have an exact sequence

$$1\to\pi_1(X_{\tilde g},\tilde *)\to\pi_1(X_g,*)\to\varphi(G)\to 1$$

where ψ is defined in [24, p. 85], so that we have a well-defined function

$$f:\mathrm{FA}^+(G,X_{\tilde g})\to \mathrm{EHom}\big(\pi_1(X_g,*),G\big)/\mathrm{Aut}^+\big(\pi_1(X_g,*)\big)$$

defined by $f(\varphi)=[\varphi^{-1}\circ\psi]$. To prove our proposition, it suffices therefore to show that the induced function

$$f^-: H^+(X_{\tilde g})\backslash\mathrm{FA}^+(G,X_{\tilde g})\to \mathrm{EHom}\big(\pi_1(X_g,*),G\big)/\mathrm{Aut}^+\big(\pi_1(X_g,*)\big)$$

is a bijection.

Let $\theta\in\mathrm{EHom}(\pi_1(X_g,*),G)$, then by [24, p. 82] $\mathrm{Ker}\,\theta = p_\#(\pi_1(X_{\tilde g},\tilde *))$ where $p:(X_{\tilde g},\tilde *)\to(X_g,*)$ is a covering projection which we may assume to be orientation preserving. Let D be the group of covering transformations of this covering, then we have a commutative diagram with exact rows

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(X_{\tilde g},\tilde *) \arrow[r] \arrow[d, equal] & \pi_1(X_g,*) \arrow[r] \arrow[d, equal] & D \arrow[r] & 1 \\
1 \arrow[r] & \pi_1(X_{\tilde g},\tilde *) \arrow[r] & \pi_1(X_g,*) \arrow[r] & G \arrow[r] \arrow[u, "\varphi"'] & 1
\end{tikzcd}$$

where $\varphi: G\to D\leqslant H^+(X_g)$ is a group isomorphism, hence $\varphi\in\mathrm{FA}^+(G,X_g)$ and $f(\varphi)=[\theta]$.

Suppose that $\varphi_1,\varphi_2\in\mathrm{FA}^+(G,X_g)$ and let p_1,p_2 be the corresponding covering projections, so that we have the exact sequences for $i=1,2$

$$1\to\pi_1(X_{\tilde g},\tilde *)\to\pi_1(X_g,*)\to\varphi_i(G)\to 1$$

with $p_{i*}(z_{\hat g})=|G|z_g$ and suppose that $\varphi_2^{-1}\circ\psi_2\circ\theta=\varphi_1^{-1}\circ\psi_1$ for some $\theta\in\mathrm{Aut}^+(\pi_1(X_g,*))$. Dehn–Nielson theorem [26, p. 194] shows that there exists $h\in H^+(X_g)$, $h(*)=*$ such that $h_\#=\theta$. Then $p_{2\#}(\pi_1(X_{\tilde g},\tilde *))=\mathrm{Ker}\,\psi_2=\theta\circ p_{1\#}(\pi_1(X_{\tilde g},\tilde *))$ and there exists $k\in H^+(X_{\tilde g})$, $k(\tilde *)=\tilde *$ such that $h\circ p_1\circ k=p_2$ by [24, p. 80] and we have $k\varphi_2(s)k^{-1}=\varphi_1(s)$ for all $s\in G$, hence f^- is injective. □

Proposition 1.3. *Let G be a finite Abelian group and let $g\geqslant 1$, then there is a one-to-one correspondence between the set*

$$\mathrm{EHom}\big(\pi_1(X_g,*),G\big)/\mathrm{Aut}^+\big(\pi_1(X_g,*)\big)\quad (\textit{respectively } \mathrm{Aut}(G)\backslash \mathrm{EHom}\big(\pi_1(X_g,*),G\big)/\mathrm{Aut}^+\big(\pi_1(X_g,*)\big))$$

and the set

$$\mathrm{EHom}\big(\mathbf{Z}^{2g},G\big)/\mathrm{Sp}_{2g}(\mathbf{Z})\quad (\textit{respectively } \mathrm{Aut}(G)\backslash \mathrm{EHom}\big(\mathbf{Z}^{2g},G\big)/\mathrm{Sp}_{2g}(\mathbf{Z})).$$

Proof. Identifying $\varphi \in \mathrm{Aut}(H_1(X_g))$ by its matrix relative to the ordered basis $(a_1, \dots, a_g, b_1, \dots, b_g)$ of $H_1(X_g)$, we observe that the naturality of the cup product

$$\cup : H^1(X_g) \times H^1(X_g) \to H^2(X_g)$$

shows that for any homeomorphism k of X_g we must have

$$k_* \in \mathrm{Sp}_{2g}^{\pm}(\mathbf{Z}) = \{A \in M_{2g}(\mathbf{Z}):\ {}^tAJA = \pm J\} \quad \text{with } J = \left[\begin{array}{c|c} & I_g \\ \hline -I_g & \end{array}\right].$$

Furthermore, Hopf formula [26, p. 98] shows that

$$k \in H^+(X_g) \quad \text{iff} \quad k_* \in \mathrm{Sp}_{2g}(\mathbf{Z}) = \{A \in M_{2g}(\mathbf{Z}):\ {}^tAJA = J\}.$$

Note also that by [18, p. 178] we have a surjective group homomorphism $H(X_g) \to \mathrm{Sp}_{2g}^{\pm}(\mathbf{Z})$ defined by $k \to k_*$.

Let $h : \pi_1(X_g, *) \to H_1(X_g)$ be the Hurwitz homomorphism, then any $\theta \in \mathrm{EHom}(\pi_1(X_g, *), G)$ induces a unique $\theta^- \in \mathrm{EHorn}(H_1(X_g), G)$ such that $\theta^- = \theta^- \circ h$ and Dehn–Nielson theorem [26, p. 194] shows that the surjective function

$$f : \mathrm{EHom}(\pi_1(X_g, *), G) \to \mathrm{EHom}(H_I(X_g), G)/\mathrm{Sp}_{2g}(\mathbf{Z})$$

defined by $f(\theta) = [\theta^-]$ induces a surjection

$$f^- : \mathrm{EHom}(\pi_1(X_g, *), G)/\mathrm{Aut}^+(\pi_1(X_g, *)) \to \mathrm{EHom}(H_1(X_g), G)/\mathrm{Sp}_{2g}(\mathbf{Z}).$$

Suppose that $\theta_i \in \mathrm{EHom}(\pi_1(X_g, *), G)$, $i = 1, 2$ such that $\theta_2^- = \theta_1^- \circ \alpha$ for some $\alpha \in \mathrm{Sp}_{2g}(\mathbf{Z})$, then $\alpha = k_*$ for some $k \in H^+(X_g)$, $k(*) = *$ and $\theta_2 = \theta_2^- \circ h = \theta_1^- \circ h \circ k_\#$. Therefore $[\theta_1] = [\theta_2]$ as desired. □

Now we can give the solution of Problem 1 in our classification problem. We shall need the following lemma.

Lemma 1.4. *Let A be a commutative ring and let $G = A/\alpha_1 \oplus \cdots \oplus A/\alpha_k$ where $\alpha_1 \subseteq \alpha_2 \subseteq \cdots \subseteq \alpha_k \neq A$, then*

$$k = \max\Big\{s:\ \bigwedge^s G \neq 0\Big\} = \text{minimum number of generators of } G.$$

Proof. The first assertion follows from [3, p. 16]. Let $r =$ minimum number of generators of G, then we have an exact sequence, $A^r \to G \to 0$ which gives an exact sequence [2, p. 78], $0 = \bigwedge^{r+1} A^r \to \bigwedge^{r+1} G \to 0$ and $\bigwedge^{r+1} G = 0$. Therefore $r = k$. □

Theorem 1.5.

(1) *Let G be any group, then*

$$\mathrm{FA}^+(G, S^2) \neq \emptyset \quad \textit{iff} \quad G \text{ is the trivial group.}$$

(2) *Let G be a finite group, then*

$$\mathrm{FA}^+(G, X_1) \neq \emptyset \quad \textit{iff} \quad G = \mathbf{Z}/(d_1) \oplus \mathbf{Z}/(d_2),\ (d_1) \subseteq (d_2).$$

(3) *Let $G = \mathbf{Z}/(d_1) \oplus \cdots \oplus \mathbf{Z}/(d_k)$, $0 \neq (d_1) \subseteq \cdots \subseteq (d_k) \neq \mathbf{Z}$ and let $\hat{g} \geqslant 2$. Then*

$$\mathrm{FA}^+(G, X_{\hat{g}}) \neq \emptyset \quad \textit{iff} \quad \text{(a) } |G| \mid (\hat{g} - 1); \quad \text{(b) } k \leqslant 2\big((\hat{g} - 1)/|G| + 1\big).$$

Proof. Part 1 is Proposition 1.1.

By Proposition 1.2, if G is a finite group, $\hat{g} \geqslant 1$, then $\mathrm{FA}^+(G, X_{\hat{g}}) \neq \emptyset$ iff $|G| \mid (\hat{g} - 1)$ and there exists an epimorphism $\pi_1(X_g, *) \to G \to 1$ where $\hat{g} - 1 = |G|(g - 1)$. Part 2 of our theorem follows by virtue of Lemma 1.4.

By Proposition 1.3, if $G = \mathbf{Z}/(d_1) \oplus \cdots \oplus \mathbf{Z}/(d_k)$, $0 \neq (d_1) \subseteq \cdots \subseteq (d_k) \neq \mathbf{Z}$ and $\hat{g} \geqslant 2$, then

$\mathrm{FA}^+(G, X_{\hat{g}}) \neq \emptyset$ iff $|G| \mid (\hat{g} - 1)$ and there exists an epimorphism $\mathbf{Z}^{2g} \to G \to 1$ where $\hat{g} - 1 = |G|(g - 1)$. Part 3 of our theorem follows by virtue of Lemma 1.4. □

The following proposition shows that it suffices to consider our classification problem only for a finite Abelian p-group for some prime p.

Proposition 1.6. *Let G be a finite Abelian group and let $\{G_i\colon 1 \leqslant i \leqslant n\}$ be the set of p-Sylow subgroups of G for all p prime $|G|$ and let $g \geqslant 1$, then there is a one-to-one correspondence between the two sets*

$$\mathrm{EHom}\big(\mathbf{Z}^{2g}, G\big)/\operatorname{Sp}_{2g}(\mathbf{Z}) \quad (\textit{respectively}\ \operatorname{Aut}(G)\backslash \mathrm{EHom}\big(\mathbf{Z}^{2g}, G)/\operatorname{Sp}_{2g}(\mathbf{Z}))$$

and

$$\prod_{i=1}^{n}\big(\mathrm{EHom}\big(\mathbf{Z}^{2g}, G_i)/\operatorname{Sp}_{2g}(\mathbf{Z})\big) \quad (\textit{respectively} \prod_{i=1}^{n}\big(\operatorname{Aut}(G_i)\backslash \mathrm{EHom}\big(\mathbf{Z}^{2g}, G_i\big)/\operatorname{Sp}_{2g}(\mathbf{Z})\big)).$$

Proof. $G = \prod_{i=1}^{n} G_i$ and if $q : G \to G_i$ is the projection onto the ith factor for $1 \leqslant i \leqslant n$, then we have a function

$$f : \mathrm{EHom}\big(\mathbf{Z}^{2g}, G\big)/\operatorname{Sp}_{2g}(\mathbf{Z}) \to \prod_{i=1}^{n}\big(\mathrm{EHom}\big(\mathbf{Z}^{2g}, G_i\big)/\operatorname{Sp}_{2g}(\mathbf{Z})\big)$$

defined by

$$f\big([\theta]\big) = \big([q_i \circ \theta]\big)_{1 \leqslant i \leqslant n}.$$

If $\operatorname{Ann}(G) = (d) = \prod_{i=1}^{n}(p_i^{\alpha_i})$ where $(p_i^{\alpha_i}) = \operatorname{Ann}(G_i)$ and if $r_i : \mathbf{Z}/(d) \to \mathbf{Z}/(p_i^{\alpha_i})$ is the canonical map, $1 \leqslant i \leqslant n$, and if we also let, by abuse of notation, $r_i : \operatorname{Sp}_{2g}(\mathbf{Z}/(d)) \to \operatorname{Sp}_{2g}(\mathbf{Z}/(p_i^{\alpha_i}))$ be the induced map, $1 \leqslant i \leqslant n$, then we have a group isomorphism [5, p. 53]

$$\operatorname{Sp}_{2g}\big(\mathbf{Z}/(d)\big) \to \prod_{i=1}^{n} \operatorname{Sp}_{2g}\big(\mathbf{Z}/\big(p_i^{\alpha_i}\big)\big)$$

defined by

$$M \to \prod_{i=1}^{n} r_i(M)$$

and since the canonical map $\mathbf{Z} \to \mathbf{Z}/(d)$ induces an epimorphism $\operatorname{Sp}_{2g}(\mathbf{Z}) \to \operatorname{Sp}_{2g}(\mathbf{Z}/(d))$ [21, p. 132], it follows easily that f is bijective. The fact that f induces a bijection

$$\operatorname{Aut}(G)\backslash \mathrm{EHom}\big(\mathbf{Z}^{2g}, G\big)/\operatorname{Sp}_{2g}(\mathbf{Z}) \to \prod_{i=1}^{n}\big(\operatorname{Aut}\big(G^i\big)\backslash \mathrm{EHom}\big(\mathbf{Z}^{2g}, G\big)/\operatorname{Sp}_{2g}(\mathbf{Z})\big)$$

follows by virtue of the group isomorphism

$$\operatorname{Aut}(G) \to \prod_{i=1}^{n} \operatorname{Aut}(G_i) \quad \text{given by} \quad \varphi \to \prod_{i=1}^{n} \varphi|_{G_i}. \quad \square$$

Now we introduce the bordism invariant of the elements of $\mathrm{FA}^{+}_{\mathrm{Diff}}(G, X_g)$, G finite group, where X_g is the closed connected orientable surface of genus $g \geqslant 1$ and with fundamental class z_g and equipped with its unique smooth structure.

Let $\omega_G = (E(G), p, B(G))$ be the universal principal G-bundle that comes from Milnor construction [14, p. 52]. Since $E(G)$ is contractible [10, p. 249] and G is totally disconnected, the exact homotopy sequence of the fibration ω_G [24, pp. 96 and 377] shows that $B(G)$ is a $K(G, 1)$ space. Observe that we have isomorphisms

$$\Omega_2(G) \underset{[6,\ \text{p. } 51]}{\longrightarrow} \Omega_2\big(B(G)\big) \underset{[6,\ \text{p. } 14]}{\overset{\mu}{\longrightarrow}} H_2\big(B(G)\big) = H_2(G)$$

where $\Omega_2(G)$ (respectively $\Omega_2(B(G))$ is the second oriented bordism group of G (respectively $B(G)$). The isomorphism μ holds, for any CW-complex, by virtue of the bordism spectral sequence [6, p. 17]. The composite isomorphism is defined as follows:

If M^2 is a closed oriented smooth surface of fundamental class z and if $\varphi \in \mathrm{FA}^+_{\mathrm{Diff}}(G, M^2)$ and if $q : M^2 \to M^2/\varphi(G)$ is the covering projection map and if $f : M^2/\varphi(G) \to B(G)$ is the classifying map of this covering, let z' be the fundamental class of $M^2/\varphi(G)$ such that $q_*(z) = |G|z'$ then the composite isomorphism above is defined by $[M^2, \varphi] \to f_*(z')$.

We have the bordism map

$$B : \mathrm{Diff}^+(X_{\hat{g}})\backslash \mathrm{FA}^+_{\mathrm{Diff}}(G, X_{\hat{g}}) \to \Omega_2(G)$$

defined by $[\varphi] \to [X_{\tilde{g}}, \varphi]$ and $[X_{\tilde{g}}, \varphi]$ is called the bordism invariant of the free action φ so that if $q : X_{\tilde{g}} \to X_{\tilde{g}}/\varphi(G)$ is the covering projection and if $f : X_{\tilde{g}}/\varphi(G) \to B(G)$ is the classifying map of this covering and if $\theta : X_{\tilde{g}}/\varphi(G) \to X_g$ is a homeomorphism such that $(\theta \circ q)_*(z_{\hat{g}}) = |G|z_g$ and $(\theta \circ q)(\tilde{*}) = *$ where $\hat{g} - 1 = |G|(g - 1)$ and if $\partial : \pi_1(B(G), f \circ q(\tilde{*})) \to \pi_0(G, 1)$ is the bijection obtained from the exact homotopy sequence of the fibration ω_G and if $i : G \to G$ is the function defined by $i(s) = s^{-1}$ for all $s \in G$, then by [14, p. 55] we get $i \circ \partial \circ (f \circ \theta^{-1})_\# = \varphi^{-1} \circ \psi$ where $\psi : \pi_1(X_g, *) \to \varphi(G)$ is defined in [24, p. 85] so that $i \circ \partial : \pi_1(B(G), f \circ q(*)) \to G$ is an isomorphism by which we identify these two groups. It follows by virtue of the bijection [24, p. 428] $[X_g, B(G)] \to G\backslash \mathrm{Hom}(\pi_1(X_g, *), G)$ defined by $h \to [h_\#]$ that the bordism map is the composite

$$\begin{aligned}\mathrm{Diff}^+(X_{\hat{g}})\backslash \mathrm{FA}^+_{\mathrm{Diff}}(G, X_{\hat{g}}) &\underset{\text{Prop. 0}}{=} H^+(X_{\hat{g}})/\mathrm{FA}^+(G, X_{\hat{g}}) \\ &\underset{\text{Prop. 1.2}}{=} \mathrm{EHom}\big(\pi_1(X_g, *), G\big)/\mathrm{Aut}^+\big(\pi_1(X_g, *)\big) \to H_2(G) = \Omega_2(G)\end{aligned}$$

where the function

$$\mathrm{EHom}\big(\pi_1(X_g, *), G\big)/\mathrm{Aut}^+\big(\pi_1(X_g, *)\big) \to H_2(G)$$

is defined by

$$[s] \to s_*(z_g).$$

The computation of the bordism invariant of the elements of $\mathrm{FA}^+_{\mathrm{Diff}}(G, X_{\tilde{g}})$ in case of a finite Abelian group G is therefore achieved by the following general theorem.

Theorem 1.7. *Let G be a finitely generated Abelian group and let $s : \pi_1(X_g, *) \to G$ be a group homomorphism then*

(1) $H_2(G) \cong \bigwedge^2 G$.
(2) $s_* : H_2(\pi_1(X_g, *)) \to H_2(G)$ *is given with respect to the isomorphism in* (1) *by* $s_*(z_g) = \sum_{i=1}^{g} s(a_i) \wedge s(b_i)$.

Proof. Let $G = \mathbf{Z}/(d_1) \oplus \cdots \oplus \mathbf{Z}/(d_n)$, $(d_1) \subseteq \cdots \subseteq (d_n)$, then (1)

$$\begin{aligned}H_2\big(\mathbf{Z}/(d_1) \oplus \cdots \oplus \mathbf{Z}/(d_n)\big) &\cong \big[H\big(\mathbf{Z}/(d_1) \oplus \cdots \oplus \mathbf{Z}/(d_{n-1})\big) \otimes H\big(\mathbf{Z}/(d_n)\big)\big]_2 \\ &\cong H_2\big(\mathbf{Z}/(d_1) \oplus \cdots \oplus \mathbf{Z}/(d_{n-1})\big) \oplus \Bigg(\bigoplus_{1 \leqslant i \leqslant n-1} \mathbf{Z}/(d_i) \otimes \mathbf{Z}/(d_n)\Bigg) \\ &\cong \cdots \cong \bigoplus_{1 \leqslant i < j \leqslant n} \big(\mathbf{Z}/(d_i) \otimes \mathbf{Z}/(d_j)\big) \cong \bigwedge^2 \Bigg(\bigoplus_{1 \leqslant i \leqslant n} \mathbf{Z}/(d_i)\Bigg)\end{aligned}$$

where the isomorphisms in the first three lines come by repeated application of Künneth Theorem [24, pp. 235 and 503] and the last isomorphism is given in [2, p. 85].

(2) Evaluate s_* on z_g by using the explicit representation of the Elienberg–Zilber chain map given in [13, p. 65], then under the isomorphisms of the first three lines in (1) above we get

$$s_*(z_g) = \sum_{k=1}^{g} \sum_{1 \leqslant i < j \leqslant n} s_i(a_k) \otimes s_j(b_k) - s_i(b_k) \otimes s_j(a_k)$$

so applying the last isomorphism in (1) above we get the desired formula. $\square$

2. Symplectic geometry over a local ring

In this section we study symplectic geometry over a local ring. We shall fix the following notation and terminology. R is a local ring with maximal ideal m_o and residue field $\mathbf{F} = R/m_0$ and let $\pi : R \to \mathbf{F}$ be the canonical map. Any R-space V is a free R-module of finite rank and $W \subseteq V$ is an R-subspace of V if W is a direct summand of V, hence W is a projective R-module and therefore it is an R-space itself [5, p. 84] and we say that W is a line (respectively plane, respectively hyperplane) if rank $W = 1$ (respectively 2, respectively $\mathrm{rank}(V) - 1$). Suppose that β is an alternating bilinear form on V (i.e. $\beta(x, x) = 0$ for all $x \in V$). We let $d_\beta : V \to V^*$ be the partial map defined by $d_\beta(y) = \beta(\cdot, y)$ and we define the hyperbolic rank of $\beta(= HR(\beta)) = \mathrm{rank}(\beta(e_i, e_j))_{1\leqslant i,j\leqslant k}$ where $\{e_i\colon\ 1 \leqslant i \leqslant k\}$ is some basis of V. Note that $HR(\beta)$ is always even [4, p. 79]. In particular, if $HR(\beta) = \mathrm{rank}(V)$ we say that (V, β) is an inner product space over R. If W is an R-subspace of V, we set $W^\perp = \{x \in V\colon\ \beta(x, y) = 0 \text{ for all } y \in W\}$. An R-space V is the orthogonal direct sum of two R-subspaces W, W' if $V = W \oplus W'$ and $W' = W^\perp$ and we write $V = W \perp W'$. We have the following lemma.

Lemma 2.1. [19, p. 151] *Let (V, β) be an inner product space over R and let W be an R-subspace of V, then*

(1) *$W^\perp$ is an R-subspace of V.*
(2) *$W^{\perp\perp} = W$.*
(3) *$W^\perp \cong (V/W)^*$ and $W^* \cong V/W^\perp$.*

Note that if $B = \{e_1, \ldots, e_k\}$ is a basis of V, then if $\mathrm{Alt}^2(V) =$ the R-module of alternating bilinear forms on V and if $U_k(R) =$ the R-module of alternating $k \times k$-matrices over

$$R = \bigl\{A \in M_k(R)\colon\ {}^tA = -A \text{ and all diagonal elements of } A \text{ are zeros}\bigr\},$$

then we have an R-module isomorphism

$$\mathrm{Alt}^2(V) \to U_k(R) \quad \text{defined by} \quad \beta \to \bigl(\beta(e_i, e_j)\bigr)_{1\leqslant i,j\leqslant k}.$$

We let $U_{k,m}(R) = \{A \in U_k(R)\colon\ \mathrm{Rank}\,\pi A = m\}$ for $0 \leqslant m \leqslant k$.

Let $B' = \{e'_1, \ldots, e'_k\}$ be the dual basis of B so that the isomorphism $V \to V^*$ induced by the function $e_i \to e'_i$ for $1 \leqslant i \leqslant k$ extends to an isomorphism $\bigwedge^2 V \to \bigwedge^2 V^*$ and composing with the isomorphism $\theta_\wedge : \bigwedge^2 V^* \to (\bigwedge^2 V)^*$ of [2, p. 153] and the isomorphism $(\bigwedge^2 V)^* \to \mathrm{Alt}^2(V)$ of [2, p. 80] and the above mentioned isomorphism, we obtain an isomorphism $X : \bigwedge^2 V \to U_k(R)$ defined by

$$\left(X\left(\sum_{1\leqslant i<j\leqslant k} x_{ij} e_i \wedge e_j\right)\right)_{st} = \begin{Bmatrix} -x_{st} & 1 \leqslant s < t \leqslant k \\ x_{ts} & 1 \leqslant t < s \leqslant k \\ 0 & 1 \leqslant s = t \leqslant k \end{Bmatrix}$$

and we define the index of $x \in \bigwedge^2 V$ by $i(x) = \frac{1}{2}\,\mathrm{Rank}\,\pi X(x)$, where π denotes also the induced map $U_k(R) \to U_k(\mathbf{F})$ by $\pi : R \to \mathbf{F}$ by abuse of notation, which is well-defined independent of the chosen basis of V.

Our first step is to obtain the Witt Decomposition Theorem, Theorem 2.5, for any alternating bilinear form on any R-space. We shall need the following lemma.

Lemma 2.2. [19, p. 151] *Let V be an R-space and let β be an alternating bilinear form on V and suppose that $(\beta(e_i, e_j))_{1\leqslant i,j\leqslant k}$ is an invertible matrix where $\{e_1, \ldots, e_k\} \subseteq V$, then $V = W \perp W'$ where $W = \bigoplus_{1\leqslant i\leqslant k} Re_i$ and where $(W, \beta|_{W\times W})$ is an inner product space over R.*

If β is an alternating bilinear form on an R-space V, then a plane $W \subseteq V$ is called a β-hyperbolic plane if $\beta(W \times W) \not\subset m_0$ hence there exists $e, f \in W$ such that $\beta(e, f) = 1$ and we must have $W = Re \oplus Rf$ by Lemma 2.2. An easy induction on $\mathrm{rank}(V)$, using Lemma 2.2, establishes the following theorem.

Theorem 2.3. *Let (V, β) be an inner product space over R, then $V = H_1 \perp H_2 \perp \cdots \perp H_t$ where $\{H_i\colon\ 1 \leqslant i \leqslant t\}$ is a set of β-hyperbolic planes of V.*

Corollary 2.4. *Let (V,β) be an inner product space over R, then there exists a basis $\{e_1,\dots,e_{2g}\}$ of V such that*

$$\big(\beta(e_i,e_j)\big)_{1\leqslant i,j\leqslant 2g}=\left[\begin{array}{c|c} & I_g \\ \hline -I_g & \end{array}\right].$$

Now we can establish the Witt Decomposition Theorem.

Theorem 2.5 *(Witt Decomposition Theorem). Let V be an R-space and let β be an alternating bilinear form on V, then there exists an R-subspace W of V such that*:

(1) $(W,\beta|_{W\times W})$ *is an inner product space over R.*
(2) $V=W\oplus W^{\perp}$.
(3) $\beta(W^{\perp}\times W^{\perp})\subseteq m_0$.

Proof. Note that the group homomorphism

$$\mathrm{Aut}_R(V)\to \mathrm{Aut}_{\mathbf{F}}(V\otimes\mathbf{F})\quad\text{defined by}\quad h\to h\otimes 1$$

is surjective by [5, pp. 82 and 89], therefore by [4, p. 79, Theorem 1] there exists an R-subspace W of V such that $\operatorname{rank} W=HR(\beta)$ and $(W,\beta|_{W\times W}))$ is an inner product space over R and $V=W\oplus W^{\perp}$ by Lemma 2.2.

If $W^{\perp}=\bigoplus_{1\leqslant i\leqslant k}Rf_i$, then $\beta(f_i,f_j)\in m_0$ for $1\leqslant i,j\leqslant k$ (since otherwise Lemma 2.2 would give $HR(\beta)>\operatorname{rank}(W)$ which is absurd). □

The following proposition is of fundamental importance.

Proposition 2.6. *Let V be an R-space of rank $2g$ and let W be an R-subspace of V of* rank k. *Suppose that (V,β) is an inner product space over R, then $HR(\beta|_{W\times W})\geqslant 2(k-g)$. Conversely, if β' is an alternating bilinear form on W such that $HR(\beta')\geqslant 2(k-g)$, then $\beta'=\beta|_{W\times W}$ where (V,β) is an inner product space over R.*

Proof. Suppose that (V,β) is an inner product space over R. We shall show that $HR(\beta|_{W\times W})\geqslant 2(k-g)$. We may assume that $k>g$.

Note that $HR(\beta|_{W\times W})=\operatorname{Rank}(d_{\beta|W\times W}\otimes 1)$ and that $d_{\beta|W\times W}\otimes 1$ is the composition $W\otimes\mathbf{F}\to V\otimes\mathbf{F}\to V^*\otimes\mathbf{F}\to W^*\otimes\mathbf{F}$ where $i:W\to V$ is the canonical injection.

We have $i\otimes 1$ injective by [2, p. 63] and ${}^t i\otimes 1$ surjective by [2, p. 58] and $d_\beta\otimes 1$ bijective since (V,β) is an inner product space over R, hence

$$\begin{aligned}HR(\beta|_{W\times W})&=\dim_{\mathbf{F}}\big(i\otimes 1(W\otimes\mathbf{F})+\operatorname{Ker}\big({}^t i d_\beta\otimes 1\big)\big)/\operatorname{Ker}\big({}^t i d_\beta\otimes 1\big)\\&=k-\dim_{\mathbf{F}}\big(i\otimes 1(W\otimes\mathbf{F})\cap\operatorname{Ker}\big({}^t i d_\beta\otimes 1\big)\big)\\&\geqslant k-\dim_{\mathbf{F}}\operatorname{Ker}\big({}^t i d_\beta\otimes 1\big)=k-(2g-k)=2(k-g).\end{aligned}$$

Conversely, suppose that β' is an alternating bilinear form on W such that $HR(\beta')\geqslant 2(k-g)$. It follows from Theorems 2.3 and 2.5 that $W=\mathbf{Z}\oplus\mathbf{Z}^{\perp}$ where $\operatorname{rank}(\mathbf{Z})=2(k-g)$ and $(\mathbf{Z},\beta'|_{W\times W})$ an inner product space over $\mathbf{Z}$.

Let $\mathbf{Z}^{\perp}=\bigoplus_{1\leqslant i\leqslant 2g-k}Re_i$ and $V=W\oplus W'$ where $W'=\bigoplus_{1\leqslant i\leqslant 2g-k}Rf_i$ and define a bilinear form β on V such that:

(1) $\beta|_{W\times W}=\beta'$;
(2) $-\beta(e_j,f_i)=\beta(f_i,e_j)=\delta_{ij}$ for $1\leqslant i,j\leqslant 2g-k$;
(3) $-\beta(x,f_i)=\beta(f_i,x)=0$ for $1\leqslant i\leqslant 2g-k$ and $x\in\mathbf{Z}\oplus W'$.

One can easily see that (V,β) is an inner product space over R as desired. □

Before stating our main theorem, we recall that if A is a commutative ring and if M, N are A-modules with M free A-module, then $\mathrm{Pol}^j(M,N)$ is the A-module of homogeneous polynomial maps of degree j from M to N [3, p. 52].

Theorem 2.7 *(Main Theorem). Let (V,β) be an inner product space over R of* rank $2g$. *Suppose there is given $r_1,\dots,r_k \in V$, $1 \leqslant k \leqslant 2g$, and $(\beta_{ij})_{1\leqslant i,j\leqslant 2g} \in U_{2g,2g}(R)$ such that*:

(i) $\sum_{1\leqslant i\leqslant k} Rr_i$ *is an R-subspace of V of* rank k.
(ii) $\beta(r_i,r_j)=\beta_{ij}$ *for* $1\leqslant i,j\leqslant k$.

Then there exists $r_{k+1},\dots,r_{2g}\in V$ such that:

(1) $\sum_{1\leqslant i\leqslant j} Rr_i$ *is an R-subspace of V of* rank j *for all* $k+1\leqslant j\leqslant 2g$.
(2) $\beta(r_i,r_j)=\beta_{ij}$ *for* $1\leqslant i,j\leqslant 2g$.
(3) *For all $k+1\leqslant j\leqslant 2g$, there exists*

$$P_j \in \mathrm{Pol}^{j-1}\big(V^{j-1},R\big) \quad \textit{and} \quad Q_j \in \sum_{q\leqslant j-1} \mathrm{Pol}^q\big(V^{j-1}\oplus R^{j-1},V\big)$$

depending only on $r_1,\dots,r_{j-1}$ and $(\beta_{jq})_{1\leqslant q<j}$ satisfying:
(a) $P_j(r_1,\dots,r_{j-1})\in R^*$.
(b) $Q_j(x,\cdot)\in\sum_{q\leqslant j-1}\mathrm{Pol}^q(R^{j-1},V)$ *for all* $x\in V^{j-1}$.
(c) $P_j(r_1,\dots,r_{j-1})r_j = Q_j((r_1,\dots,r_{j-1}),(\beta_{jq})_{1\leqslant q<j})$.

Proof. By Corollary 2.4 we may assume that there exists an ordered basis $B=(a_1,\dots,a_g,b_1,\dots,b_g)$ of V such that β is defined on $B\times B$ by $\beta(a_i,b_i)=-\beta(b_i,a_i)=1$ for $1\leqslant i\leqslant g$ and is zero otherwise. Let β' be the canonical extension of β to $V\otimes\mathbf{F}$. By induction, we need only to obtain $r_{k+1}\in V$ so that conditions (1), (2), (3) are verified for $j=k+1$.

The $r_1,\dots,r_k$ elements of V can be expressed in a $k\times 2g$ matrix M_k over R where the row i gives the components of the element r_i with respect to the ordered basis B for $1\leqslant i\leqslant k$. Let $W=\sum_{1\leqslant i\leqslant k}Rr_i$. Since W is an R subspace of V of rank k, there must exist by [5, p. 84] and [2, p. 88, Proposition 12 and p. 96, Proposition 9] column vectors of M_k, $c_{i_1},c_{i_2},\dots,c_{i_m},c_{j_1},c_{j_2},\dots,c_{j_n}$ where $1\leqslant i_1<\dots<i_m\leqslant g$ and $g+1\leqslant j_1<\dots<j_n\leqslant 2g$ with $m+n=k$ such that if $X_k=X(c_{i_1},c_{i_2},\dots,c_{i_m},c_{j_1},c_{j_2},\dots,c_{j_n})$ is the $k\times k$-matrix over R with columns $c_{i_1},c_{i_2},\dots,c_{i_m},c_{j_1},c_{j_2},\dots,c_{j_n}$ respectively, then $\det(X_k)\notin m_0$ and X_k is invertible by [2, p. 93, Proposition 5]. Define

$$(t_{j_1},\dots,t_{j_n},-t_{i_1},\dots,-t_{i_m}) = (-\beta_{k+1,1},\dots,-\beta_{k+1,k})^t\big(X_k^{-1}\big)$$

and

$$\begin{aligned}&\big(t^i_{j_1},\dots,t^i_{j_n},-t^i_{i_1},\dots,-t^i_{i_m}\big)\\ &\quad=\left\{\begin{matrix}(t_{j_1},\dots,t_{j_n},-t_{i_1},\dots,-t_{i_m})+{}^t(X_k^{-1}c_{g+i}) & \text{for } i\in[1,g]\backslash\{i_1,\dots,i_m\}\\ (t_{j_1},\dots,t_{j_n},-t_{i_1},\dots,-t_{i_m})-{}^t(X_k^{-1}c_{i-g}) & \text{for } i\in[g+1,2g]\backslash\{j_1,\dots,j_m\}\end{matrix}\right\}.\end{aligned}$$

Set

$$v_0=\sum_{s=1}^m t_{i_s}a_{i_s}+\sum_{s=1}^n t_{j_s}b_{j_s}$$

and

$$v_i=a_i+\sum_{s=1}^m t^i_{i_s}a_{i_s}+\sum_{s=1}^n t^i_{j_s}b_{j_s} \quad \text{for } i\in[1,g]\backslash\{i_l,\dots,i_m\}$$

and

$$w_j=b_{j-g}+\sum_{s=1}^m t^j_{i_s}a_{i_s}+\sum_{s=1}^n t^j_{j_s}b_{j_s} \quad \text{for } j\in[g+1,2g]\backslash\{j_1,\dots,j_n\}$$

then $\beta(r_s,r_0)=\beta(r_s,r_i)=\beta(r_s,w_j)=\beta_{s,k+1}$ for $1\leqslant s\leqslant k$, $i\in[1,g]\backslash\{i_l,\dots,i_m\}$, $j\in[g+1,2g]\backslash\{j_1,\dots,j_n\}$.

Note that $W^\perp=\sum_i R(v_i-v_0)+\sum_j R(w_j-v_0)$ by [5, p. 84], Lemma 2.1.

Claim. $\{v_0 \otimes 1, v_i \otimes 1, w_j \otimes 1\colon\ i \in [1, g]\backslash\{i_l, \dots, i_m\},\ j \in [g+1, 2g]\backslash\{j_1, \dots, j_n\}\} \not\subset W \otimes \mathbf{F}$.

Proof. *Case* 1. $1 \leqslant k < g$: We have $\dim_{\mathbf{F}} \mathbf{F}v_0 \otimes 1 + \sum_i \mathbf{F}v_i \otimes 1 + \sum_j \mathbf{F}w_j \otimes 1 \geqslant 2g - k > k = \dim_{\mathbf{F}} W \otimes \mathbf{F}$, hence our claim follows in this case.

By virtue of Proposition 2.6 we have only two more cases to consider:

Case 2. $g \leqslant k$ and $\operatorname{rank}(\pi(\beta_{ij})_{1\leqslant i,j\leqslant k}) > 2(k-g)$:

If our claim fails in this case, then we must have $W^{\perp} \otimes \mathbf{F} \underset{\text{Lemma 2.1}}{=} (W \otimes \mathbf{F})^{\perp} \subseteq W \otimes \mathbf{F}$ and by Witt Decomposition Theorem, Theorem 2.5, we must have $\operatorname{rank}(\pi(\beta_{ij})_{1\leqslant i,j\leqslant k}) = k - (2g - k)$ which is absurd.

Case 3. $g \leqslant k$ and $\operatorname{rank}(\pi(\beta_{ij})_{1\leqslant i,j\leqslant k}) = 2(k-g)$: In this case we must have $v_0 \otimes 1 \notin W \otimes \mathbf{F}$ since otherwise we would have

$$\begin{aligned}\operatorname{rank}\bigl(\pi(\beta_{ij})_{1\leqslant i,j\leqslant k+1}\bigr) &= \operatorname{rank}\bigl(\pi\bigl(\beta(r_i, r_j)\bigr)_{1\leqslant i,j\leqslant k+1}\bigr), \quad \text{where } r_{k+1} = v_0\\ &= \operatorname{rank}\bigl(\bigl(\beta'(r_i \otimes 1, r_j \otimes 1)\bigr)_{1\leqslant i,j\leqslant k+1}\bigr)\\ &= \operatorname{rank}\bigl(\pi(\beta_{ij})_{1\leqslant i,j\leqslant k}\bigr) = 2(k-g) < 2(k+1-g)\end{aligned}$$

which contradicts Proposition 2.6. □

By virtue of this claim we may define

$$r_{k+1} \in \bigl\{v_0, v_i, w_j\colon\ i \in [1, g]\backslash\{i_l, \dots, i_m\},\ j \in [g+1, 2g]\backslash\{j_1, \dots, j_n\}\bigr\}$$

such that $r_{k+1} \otimes 1 \notin W \otimes \mathbf{F}$ hence $\sum_{i=1}^{k+1} Rr_i$ is an R-subspace of V of rank $k+1$ by [5, p. 84]. It follows from the above construction that there exists

$$P_{k+1} \in \operatorname{Pol}^k\bigl(V^k, R\bigr) \quad \text{and} \quad Q_{k+1} \in \sum_{q\leqslant k} \operatorname{Pol}^q\bigl(V^k \oplus R^k, V\bigr)$$

such that

$$P_{k+1}(r_l, \dots, r_k) = \det X_k \quad \text{and} \quad Q_{k+1}\bigl((r_1, \dots, r_k), (\beta_{k+1,q})_{1\leqslant q\leqslant k}\bigr) = \det X_k.r_{k+1}$$

and $Q_{k+1}(x, \cdot) \in \sum_{q\leqslant k} \operatorname{Pol}^q(R^k, V)$ for all $x \in V^k$. □

Recall that if A is a commutative ring and if β (respectively β') is a bilinear form on the A-module V (respectively V'), then $f \in \operatorname{Hom}(V, V')$ is a metric homomorphism if $\beta'(f(x), f(y)) = \beta(x, y)$ for all $x, y \in V$.

Corollary 2.8 *(Witt Theorem). Let (V, β) be an inner product space over a local ring R of* rank $2g$. *Let W, W_1 be two R-subspaces of V each of* rank k *and let $\varphi : (W, \beta|_{W\times W}) \to (W_1, \beta|_{W_1\times W_1})$ be a metric isomorphism, then there exists $\Phi : (V, \beta) \to (V, \beta)$ metric isomorphism such that $\Phi|_W = \varphi$.*

Proof. Let

$$W = \sum_{i=1}^{k} Rr_i \quad \text{and} \quad V = W \oplus \sum_{i=1}^{2g-k} Rr_{k+i}$$

and note that $(\beta(r_i, r_j))_{1\leqslant i,j\leqslant 2g} \in U_{2g,2g}(R)$. Since $\beta(\varphi r_i, \varphi r_j) = \beta(r_i, r_j)$ for $1 \leqslant i,\ j \leqslant k$, Theorem 2.7 shows that there exists $r'_{k+i} \in V$, $1 \leqslant i \leqslant 2g-k$, such that $V = \sum_{i=1}^{k} R\varphi r_i \oplus \sum_{i=1}^{2g-k} Rr'_{k+i}$ and $\beta(r'_i, r'_j) = \beta(r_i, r_j)$ for $1 \leqslant i, j \leqslant 2g$ where $r'_i = \varphi r_i$ for $1 \leqslant i \leqslant k$, hence the homomorphism $\Phi : V \to V$ defined by $\Phi(r_i) = r'_i$ for $1 \leqslant i \leqslant 2g$ is a metric isomorphism of (V, β) such that $\Phi|_W = \varphi$. □

Our main theorem, Theorem 2.7, enables us to obtain a complete description of the set $\operatorname{EHom}(V, R^k)/\operatorname{Sp}_{2g}(R)$, where V is an R space of rank $2g$ and $\operatorname{EHom}(V, R^k)$ is the set of epimorphisms from V onto R^k. The group $\operatorname{Sp}_{2g}(R)$ acts to the right on $\operatorname{EHom}(V, R^k)$ by fixing an ordered basis $B = (a_1, \dots, a_g, b_1, \dots, b_g)$ of V such that if $M \in \operatorname{Sp}_{2g}(R)$ we let f_M be the unique automorphism of V whose matrix with respect to the basis B is M then if

$\varphi \in \mathrm{EHom}(V, R^k)$ we let $\varphi.M = \varphi \circ f_M$. Note that $\mathrm{EHom}(V, R^k)/\operatorname{Sp}_{2g}(R)$ is independent of the chosen basis of V which is therefore assumed fixed.

Corollary 2.9. *Let V be an R-space of* rank $2g$ *and let $B = (a_1, \dots, a_g, b_1, \dots, b_g)$ be an ordered basis of V, then we have a bijection*

$$s : \mathrm{EHom}(V, R^k)/\operatorname{Sp}_{2g}(R) \to \left\{x \in \bigwedge^2 R^k:\ i(x) \geqslant k - g\right\}$$

defined by

$$s([f]) = \sum_{i=1}^{g} f(a_i) \wedge f(b_i).$$

Moreover, if $x = \sum_{1 \leqslant i<j \leqslant k} c_{ij} e_i \wedge e_j \in \bigwedge^2 R^k$ with $i(x) \geqslant k - g$ where $E = \{e_i:\ 1 \leqslant i \leqslant k\}$ is the canonical basis of R^k, then there exists $f \in \mathrm{EHom}(V, R^k)$ such that $s([f]) = x$ and ${}^t f$ is given with respect to the dual bases B^* and E^* of V^* and $(R^k)^*$ respectively by

$${}^t f(e_j^*) = a_j^* + \sum_{1 \leqslant i<j} c_{ij} b_i^* \quad \text{for } 1 \leqslant j \leqslant g$$

and

$$P_j\bigl({}^t f(e_1^*), \dots, {}^t f(e_{j-1}^*)\bigr) {}^t f(e_j^*) = Q_j\bigl(\bigl({}^t f(e_1^*), \dots, {}^t f(e_{j-1}^*)\bigr), (c_{qj})_{1 \leqslant q<j}\bigr) \quad \text{for } g < j \leqslant k$$

where

$$P_j \in \mathrm{Pol}^{j-1}\bigl((V^*)^{j-1}, R\bigr) \quad \text{and} \quad Q_j \in \sum_{q \leqslant j-1} \mathrm{Pol}^q\bigl((V^*)^{j-1} \oplus R^{j-1}, V^*\bigr)$$

depending only on ${}^t f(e_1^*), \dots, {}^t f(e_{j-1}^*)$ and $(c_{qj})_{1 \leqslant q<j}$ such that:

(a) $P_j({}^t f(e_1^*), \dots, {}^t f(e_{j-1}^*)) \in R^*$.
(b) $Q_j(x, .) \in \sum_{q \leqslant j-1} \mathrm{Pol}^q(R^{j-1}, V)$ for all $x \in (V^*)^{j-1}$.

Proof. Note that $\sum_{i=1}^{g} a_i \wedge b_i$ is stable under $\operatorname{Sp}_{2g}(R)$ so that the function s is well-defined. Let β be the bilinear form on V defined on $B \times B$ by $\beta(a_i, b_i) = -\beta(b_i, a_i) = 1$, $1 \leqslant i \leqslant g$ and zero otherwise, then (V, β) is an inner product space over R and $d_\beta : V \to V^*$ is an R-isomorphism so that the inverse form $\beta^\wedge$ on V^* is defined on $B^* \times B^*$ by $\beta^\wedge(a_i^*, b_i^*) = -\beta^\wedge(b_i^*, a_i^*) = -1$ for $1 \leqslant i \leqslant g$ and is zero otherwise [4, p. 23] and $(V^*, \beta^\wedge)$ is again an inner product space over R of rank $2g$.

Let $f \in \mathrm{EHom}(V, R^k)$ and let $M(f) = (f_{ij})_{1 \leqslant i \leqslant k, 1 \leqslant j \leqslant 2g}$ be the matrix of f with respect to the bases B and E of V and R^k respectively, then $M({}^t f) = {}^t(M(f))$ by [2, p. 145]. Note that ${}^{tt} f = f$ by [2, p. 47] shows that ${}^t f : (R^k)^* \to V^*$ is an isomorphism onto a direct factor of V^* by [5, p. 85] so that by [2, p. 37] we have a bijection

$$\theta : \mathrm{EHom}(V, R^k) \to \left\{(r_1, \dots, r_k) \in (V^*)^k:\ \sum_{i=1}^{k} R r_i \text{ is an } R\text{-subspace of } V^* \text{ of rank}\, k\right\}$$

defined by

$$\theta(f) = \bigl({}^t f(e_i^*)\bigr)_{1 \leqslant i \leqslant k}.$$

Note that $\operatorname{Sp}_{2g}(R)$ acts to the right on these two sets so that the function θ induces a bijection

$$\mathrm{EHom}(V, R^k)/\operatorname{Sp}_{2g}(R) \to \left\{(r_1, \dots, r_k) \in (V^*)^k:\ \sum_{i=1}^{k} R r_i \text{ is an } R\text{-subspace of } V^* \text{ of rank}\, k\right\} \Big/ \operatorname{Sp}_{2g}(R)$$

by which we identify these two sets. We have

$$\sum_{t=1}^{g} f(a_t)\wedge f(b_t)=\sum_{t=1}^{g}\sum_{1\leqslant i,j\leqslant k} f_{i,t}f_{j,g+t}e_i\wedge e_j=\sum_{1\leqslant i<j\leqslant k}\sum_{t=1}^{g}(f_{i,t}f_{j,g+t}-f_{i,g+t}f_{j,t})e_i\wedge e_j$$
$$=-\sum_{1\leqslant i<j\leqslant k}\left\langle\sum_{p=1}^{g} f_{j,p}b_p-f_{j,g+p}a_p,\ \sum_{q=1}^{g} f_{i,q}a_q^*+f_{i,g+q}b_q^*\right\rangle e_i\wedge e_j$$
$$=-\sum_{1\leqslant i<j\leqslant k}\langle d_\beta^{-1}\big({}^t f(e_j^*)\big),{}^t f(e_i^*)\rangle e_i\wedge e_j=-\sum_{1\leqslant i<j\leqslant k}\beta^\wedge\big({}^t f(e_i^*),{}^t f(e_j^*)\big)e_i\wedge e_j.$$

Therefore, to show that the function s is bijective it suffices to show that the function

$$\left\{(r_1,\dots,r_k)\in(V^*)^k\colon\ \sum_{i=1}^{k}Rr_i \text{ is an } R\text{-subspace of } V^* \text{ of rank}\,k\right\}\Big/\operatorname{Sp}_{2g}(R)\to\left\{x\in\bigwedge^2 R^k\colon\ i(x)\geqslant k-g\right\}$$

defined by

$$\big[(r_1,\dots,r_k)\big]\to-\sum_{1\leqslant i<j\leqslant k}\beta^\wedge(r_i,r_j)e_i\wedge e_j$$

is bijective. Injectivity follows from Witt Theorem, Corollary 2.8, and surjectivity follows from Proposition 2.6 and Corollary 2.4.

Suppose that

$$x=\sum_{1\leqslant i<j\leqslant k}c_{ij}e_i\wedge e_j\in\bigwedge^2 R^k$$

with $i(x)\geqslant k-g$, then by Proposition 2.6, there exists $(\beta_{ij})\in U_{2g,2g}(R)$ such that $\beta_{ij}=c_{ij}$ for $1\leqslant i<j\leqslant k$.

Now the required $f\in\operatorname{EHom}(V,R^k)$ is defined as follows:

For $1\leqslant j\leqslant\min(g,k)$, let ${}^t f(e_j^*)=a_j^*+\sum_{1\leqslant i<j}c_{ij}b_i^*$ then $\beta({}^t f(e_i^*),{}^t f(e_j^*))=-c_{ij}$ for $1\leqslant i<j\leqslant\min(g,k)$ and clearly $\sum_{j=1}^{\min(g,k)}R^t f(e_j^*)$ is an R-subspace of V^* of rank $\min(k,g)$. If $k>g$ we apply Theorem 2.7 to define ${}^t f(e_j^*)$ for $g<j\leqslant k$ by induction on j and the remaining assertions of our corollary follow from Theorem 2.7. □

We apply the preceding analysis to the following problem:

Assumption 2.10. A commutative ring, m maximal ideal of A, V free A-module of rank $2g$ having an ordered basis

$$B=(a_1,\dots,a_g,b_1,\dots,b_g)\quad\text{and}\quad G=A/m^{\alpha_1}\oplus\cdots\oplus A/m^{\alpha_k}\quad\text{where }\alpha_l\geqslant\cdots\geqslant\alpha_k>0.$$

Let $q_j: A/m^{\alpha_1}\to A/m^{\alpha_j}$, $1\leqslant j\leqslant k$, be the canonical epimorphisms and set $q=\bigoplus_{j=1}^k q_j$ and let $E=\{e_1,\dots,e_k\}$ be the canonical basis of $(A/m^{\alpha_1})^k$ and set $z_i=q(e_i)$, $1\leqslant i\leqslant k$.

We wish to obtain a complete description of the set $\operatorname{EHom}(V,G)/\operatorname{Sp}_{2g}(A)$ where $\operatorname{Sp}_{2g}(A)$ acts to the right on $\operatorname{EHom}(V,G)$, the set of epimorphisms from V to G, by fixing the basis B of V as before.

We need three lemmas.

Lemma 2.11. *Subject to Assumption* 2.10*, the function*

$$\operatorname{Hom}(1_V,q):\operatorname{EHom}\big(V,\big(A/m^{\alpha_1}\big)^k\big)\to\operatorname{EHom}(V,G)$$

defined by

$$s\to q\circ s$$

is surjective.

Proof. Suppose $s' \in \mathrm{EHom}(V, G)$, then $s' = q \circ s$ for some $s \in \mathrm{Hom}(V, (A/m^{\alpha_1})^k)$ since q is onto and V is a free A-module. Since $q \otimes 1 : (A/m^{\alpha_1})^k \otimes A/m \to G \otimes A/m$ is an isomorphism, then $s \otimes 1 : V \otimes A/m \to (A/m^{\alpha_1})^k \otimes A/m$ is onto. It follows that $\{s(a_i) \otimes 1, s(b_i) \otimes 1\colon\ 1 \leqslant i \leqslant g\}$ generates $(A/m^{\alpha_1})^k \otimes A/m$ and since A/m^{α_1} is a local ring [5, p. 52], then $\{s(a_i), s(b_i)\colon\ 1 \leqslant i \leqslant g\}$ generates $(A/m^{\alpha_1})^k$ by [5, p. 83] and s is onto. $\square$

Lemma 2.12. *Subject to Assumption* 2.10*, the function*

$$\mathrm{EHom}\big(V, \big(A/m^{\alpha_1}\big)^k\big) \to \mathrm{EHom}\big(V \otimes A/m^{\alpha_1}, \big(A/m^{\alpha_1}\big)^k\big)$$

defined by

$$s \to s \otimes 1_{A/m^{\alpha_1}}$$

is bijective and induces a bijection

$$\mathrm{EHom}\big(V, \big(A/m^{\alpha_1}\big)^k\big)/\,\mathrm{Sp}_{2g}(A) \to \mathrm{EHom}\big(V \otimes A/m^{\alpha_1}, \big(A/m^{\alpha_1}\big)^k\big)/\,\mathrm{Sp}_{2g}\big(A/m^{\alpha_1}\big)$$

by which we identify these two sets.

Proof. Let $p : A \to A/m^{\alpha_1}$ be the canonical map. The fact that the first map of sets is bijective is clear since $s \otimes 1_{A/m^{\alpha_1}} \circ 1_V \otimes p = s$. To establish the fact that it induces a bijection on the corresponding quotient sets we need only to show that the group homomorphism $\mathrm{Sp}_{2g}(A) \to \mathrm{Sp}_{2g}(A/m^{\alpha_1})$ induced by p is onto.

Let β be the bilinear form on V defined on $B \times B$ by $\beta(a_i, b_i) = -\beta(b_i, a_i) = 1$ for $1 \leqslant i \leqslant g$ and zero otherwise and let β' be its canonical extension to $V \otimes A/m^{\alpha_1}$, then we have isomorphisms $\mathrm{Aut}_A(V, \beta) \to \mathrm{Sp}_{2g}(A)$ and $\mathrm{Aut}_{A/m^{\alpha_1}}(V \otimes A/m^{\alpha_1}, \beta') \to \mathrm{Sp}_{2g}(A/m^{\alpha_1})$ by considering the matrix of an automorphism with respect to the basis B and $B \otimes 1$ of V and $V \otimes A/m^{\alpha_1}$, respectively. Therefore, it suffices to show that the map

$$\mathrm{Aut}_A(V, \beta) \to \mathrm{Aut}_{A/m^{\alpha_1}}\big(V \otimes A/m^{\alpha_1}, \beta'\big)$$

given by

$$f \to f \otimes 1$$

is onto. Since A/m^{α_1} is a local ring [5, p. 52], then $\mathrm{Sp}_{2g}(A/m^{\alpha_1})$ is generated by symplectic transvections with respect to β' [19, p. 199] and these can be realized by elements of $\mathrm{Sp}_{2g}(A)$, hence the surjectivity follows. $\square$

Lemma 2.13. *Subject to Assumption* 2.10*, suppose that* $x_k = \sum_{1 \leqslant i<j \leqslant k} c^k_{ij} e_i \wedge e_j \in \bigwedge^2 (A/m^{\alpha_1})^k$*,* $i(x_k) \geqslant k - g$*,* $h = 1, 2$*, and* $q_j(c^1_{ij}) = q_j(c^2_{ij})$ *for* $1 \leqslant i < j \leqslant k$ *and let*

$$s : \mathrm{EHom}\big(V \otimes A/m^{\alpha_1}, \big(A/m^{\alpha_1}\big)^k\big) \to \bigwedge^2 \big(A/m^{\alpha_1}\big)^k$$

be the function

$$f \to \sum_{i=1}^{g} f(a_i) \wedge f(b_i)$$

(*where* $\mathrm{EHom}(V \otimes A/m^{\alpha_1}, (A/m^{\alpha_1})^k)$ *is identified with* $\mathrm{EHom}(V, (A/m^{\alpha_1})^k)$ *by the bijection of Lemma* 2.12) *then there exists* $f^h \in \mathrm{EHom}(V \otimes A/m^{\alpha_1}, (A/m^{\alpha_1})^k)$*,* $h = 1, 2$*, whose matrix* $M(f^h) = (f_{ij})_{1 \leqslant i \leqslant k, 1 \leqslant j \leqslant 2g}$ *with respect to the bases* $B \otimes 1$ *and* E *of* $V \otimes A/m^{\alpha_1}$ *and* $(A/m^{\alpha_1})^k$ *respectively have the form*

$$\begin{bmatrix} 1 & & & & 0 & & & & \\ & \bullet & & & * & \bullet & & & \\ & & \bullet & & \bullet & \bullet & \bullet & & \\ & & & \bullet & \bullet & \bullet & \bullet & \bullet & \\ & & & 1 & * & \bullet & \bullet & * & 0 \end{bmatrix} \quad \textit{if } k \leqslant g$$

and

$$\begin{bmatrix} 1 & & & & & 0 & & & & \\ & \bullet & & & & * & \bullet & & & \\ & & \bullet & & & \bullet & \bullet & \bullet & & \\ & & & \bullet & & \bullet & \bullet & \bullet & \bullet & \\ & & & & 1 & * & \bullet & \bullet & * & 0 \\ * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * \end{bmatrix} \quad \text{if } k > g$$

and such that $s(f^h) = x^h$, $h = 1, 2$, *and* $q_j(f^1_{ij}) = q_j(f^2_{ij})$ *for* $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant 2g$.

Proof. The existence of the epimorphisms f^h, $h = 1, 2$, under the hypotheses of the lemma follows directly by applying the construction of Corollary 2.9 and observing that

$$q_j(c^1_{ij}) = q_j(c^2_{ij}) \quad \text{for } 1 \leqslant i < j \leqslant k$$

gives

$$q_j(f^1_{ij}) = q_j(f^2_{ij}) \quad \text{for } 1 \leqslant i \leqslant k,\ 1 \leqslant j \leqslant 2g. \quad \square$$

Now we can obtain the solution of our problem and get a complete description of $\mathrm{EHom}(V, G)/\mathrm{Sp}_{2g}(A)$ under Assumption 2.10.

Theorem 2.14. *Subject to Assumption* 2.10, *the function* $s' : \mathrm{EHom}(V, G) \to \bigwedge^2 G$ *defined by*

$$f \to \sum_{i=1}^{g} f(a_i) \wedge f(b_i)$$

induces a bijection, also denoted by s', *by abuse of notation,*

$$s' : \mathrm{EHom}(V, G)/\mathrm{Sp}_{2g}(A) \to \left\{ x \in \bigwedge^2 G \colon i\left(\left(\bigwedge^2 q\right)^{-1}(x)\right) \geqslant k - g \right\}.$$

Proof. Observe that we have a commutative diagram

$$\begin{tikzcd} \mathrm{EHom}(V, (A/m^{\alpha_1})^k)/\mathrm{Sp}_{2g}(A) \arrow[r, "\mathrm{Hom}(1_V, q)", "\text{onto, Lemma 2.11}"'] \arrow[d, equal, "\text{Lemma 2.12}"] & \mathrm{EHom}(V, G)/\mathrm{Sp}_{2g}(A) \arrow[dd, "s'"'] \\ \mathrm{EHom}(V \otimes A/m^{\alpha_1}, (A/m^{\alpha_1})^k)/\mathrm{Sp}_{2g}(A/m^{\alpha_1}) \arrow[d, "s"', "\text{bijection Corollary 2.9}"] & \\ \{x \in \bigwedge^2 (A/m^{\alpha_1})^k \colon i(x) \geqslant k - g\} \arrow[r, "\bigwedge^2 q", "\text{onto [2, p. 78]}"'] & \{x \in \bigwedge^2 G \colon i((\bigwedge^2 q)^{-1}(x)) \geqslant k - g\} \end{tikzcd}$$

The surjectivity of the map s' is therefore clear from the above diagram. Note that if $f \in \mathrm{EHom}(V \otimes A/m^{\alpha_1}, (A/m^{\alpha_1})^k)$ has a matrix representation $(f_{ij})_{1 \leqslant i \leqslant k,\ 1 \leqslant j \leqslant 2g}$ with respect to the bases $B \otimes 1$ and E of $V \otimes A/m^{\alpha_1}$ and $(A/m^{\alpha_1})^k$ respectively, then under the map of the top row in the above diagram f defines the element $f^- \in \mathrm{EHom}(V, G)$ given by

$$f^-(a_j) = \sum_{i=1}^{k} q_i(f_{ij}) z_i \quad \text{and} \quad f^-(b_j) = \sum_{i=1}^{k} q_i(f_{i,g+j}) z_i$$

for $1 \leqslant j \leqslant g$. To prove the injectivity of the map s' of the theorem, let $f^h \in \mathrm{EHom}(V \otimes A/m^{\alpha_1}, (A/m^{\alpha_1})^k)$, $h = 1, 2$ such that $s(f^h) = \sum_{1 \leqslant i < j \leqslant k} c^k_{ij} e_i \wedge e_j \in \bigwedge^2 (A/m^{\alpha_1})^k$ and $i(s(f^h)) \geqslant k - g$ for $h = 1, 2$ and $q_j(c^1_{ij}) = q_j(c^2_{ij})$ for $1 \leqslant i < j \leqslant k$. It follows by Lemma 2.13 and Corollary 2.9 that we may assume that the matrix representation

$(f^h_{ij})_{1\leqslant i\leqslant k,\ 1\leqslant j\leqslant 2g}$ of f^h, $h=1,2$, with respect to the bases $B\otimes 1$ and E of $V\otimes A/m^{\alpha_1}$ and $(A/m^{\alpha_1})^k$ respectively satisfies $q_j(f^1_{ij})=q_j(f^2_{ij})$ for $1\leqslant i\leqslant k$, $1\leqslant j\leqslant 2g$. Therefore $[f^{1-}]=[f^{2-}]$ in $\mathrm{EHom}(V,G)/\mathrm{Sp}_{2g}(A)$ as desired. □

Corollary 2.15 *(Zimmermann normal form).* [27] *Subject to Assumption* 2.10*, suppose that* $1\leqslant k\leqslant g$ *then a complete system of representatives of the orbits of* $\mathrm{EHom}(V,G)$ *under the action of* $\mathrm{Sp}_{2g}(A)$ *is*

$$S(G)=\left\{f\in \mathrm{EHom}(V,G)\colon\ f(a_j)=\begin{Bmatrix} z_j & 1\leqslant j\leqslant k\\ 0 & \textit{otherwise}\end{Bmatrix}\ \textit{and}\right.$$
$$\left. f(b_j)=\begin{Bmatrix}\sum_{j<i\leqslant k} c_{ij}z_i & 1\leqslant j\leqslant k\ (c_{ij}\in A/m^{\alpha_i},\ j<i\leqslant k)\\ 0 & \textit{otherwise}\end{Bmatrix}\right\}.$$

Proof. It follows by Lemma 2.13 and the diagram of Theorem 2.14 that every orbit of $\mathrm{Sp}_{2g}(A)$ on $\mathrm{EHom}(V,G)$ contains an element of $S(G)$. Note that the map s' of Theorem 2.14 gives the map $S(G)\to\bigwedge^2 G$ defined by

$$f\to\sum_{j=1}^{k} z_j\wedge\sum_{j<i\leqslant k} c_{ij}z_i=\sum_{1\leqslant i<j\leqslant k} c_{ji}z_i\wedge z_j$$

which is injective and therefore each orbit of $\mathrm{Sp}_{2g}(A)$ intersects $S(G)$ in a unique point. □

3. Equivalence and weak equivalence classes of finite Abelian group actions

In this section we obtain the complete solution of Problem 2 of our classification problem for a finite Abelian group G and we also obtain the solution of Problem 3 of that problem when all the p-Sylow subgroups of G are homocyclic (i.e. of the form $(\mathbf{Z}/p^\alpha\mathbf{Z})^n$). We shall need the following lemmas.

Lemma 3.1.

$$\left|U_{k,2m}(\mathbf{F}_q)\right|=q^{m(2(k-m)-1)}\frac{\prod_{i=1}^{2m}(1-q^{-(k-2m+i)})}{\prod_{i=1}^{m}(1-q^{-2i})}.$$

where $U_{k,2m}(\mathbf{F}_q)=\{A\in U_k(\mathbf{F}_q)\colon\ \mathrm{Rank}\,\pi A=2m\}$ *for* $0\leqslant 2m\leqslant k$.

Proof. Note that by [4, p. 79, Theorem 1], we have

$$\left|U_{k,2m}(\mathbf{F}_q)\right|=\left|GL_k(\mathbf{F}_q)\colon\ \mathrm{Sp}_k(\Phi_m)\cap GL_k(\mathbf{F}_q)\right|$$

where

$$\mathrm{Sp}_k(\Phi_m)=\left\{A\in M_k(\mathbf{F}_q)\colon\ {}^tAJ_{k,m}A=J_{k,m}\right\}$$

and

$$J_{k,m}=\begin{bmatrix} 0 & I_m & \\ -I_m & 0 & \\ & & 0\end{bmatrix}\in M_k(\mathbf{F}_q).$$

One can easily see that

$$\mathrm{Sp}_k(\Phi_m)=\left\{\begin{bmatrix} A & 0\\ * & *\end{bmatrix}\in M_k(\mathbf{F}_q)\colon\ A\in\mathrm{Sp}_{2m}(\mathbf{F}_q)\right\}.$$

We have by [9, pp. 9 and 19]

$$\left|GL_k(\mathbf{F}_q)\right|=q^{k^2}\prod_{i=1}^{k}\left(1-q^{-i}\right)\quad\text{and}\quad\left|\mathrm{Sp}_{2m}(\mathbf{F}_q)\right|=q^{2m^2+m}\prod_{i=1}^{m}\left(1-q^{-2i}\right),$$

therefore

$$\left|U_{k,2m}(\mathbf{F}_q)\right| = \frac{GL_k(\mathbf{F}_q)}{|\operatorname{Sp}_{2m}(\mathbf{F}_q)||GL_{k-2m}(\mathbf{F}_q)|q^{2m(k-2m)}} = q^{m(2(k-m)-1)}\frac{\prod_{i=1}^{2m}(1-q^{-(k-2m+i)})}{\prod_{i=1}^{m}(1-q^{-2i})}. \quad \square$$

Lemma 3.2. *Let G be a finite Abelian p-group and suppose that $(p^{\alpha_1}) = \operatorname{Ann}(G)$ and $k =$ the minimum number of generators of G and let $q : (\mathbf{Z}/(p^{\alpha_1}))^k \to G$ be any epimorphism, then*

$$\left|\left\{x \in \bigwedge^2 G\colon\ i\left(\left(\bigwedge^2 q\right)^{-1}(x)\right) \geqslant k-g\right\}\right| = \begin{Bmatrix} |\bigwedge^2 G| & k \leqslant g \\ |\bigwedge^2 G|.\sum_{m\geqslant k-g} p^{-\binom{k-2m}{2}}.\frac{\prod_{i=1}^{2m}(1-q^{-(k-2m+i)})}{\prod_{i=1}^{m}(1-q^{-2i})} & k > g \end{Bmatrix}.$$

Proof. If $k \leqslant g$ our formula is trivial, therefore we assume that $k > g$. We may assume that $G = \mathbf{Z}/(p^{\alpha_1}) \oplus \cdots \oplus \mathbf{Z}/(p^{\alpha_k})$ where $\alpha_1 \geqslant \cdots \geqslant \alpha_k > 0$ and that $q = \bigoplus_{i=1}^k q_i$ where $q_i : \mathbf{Z}/(p^{\alpha_1}) \to \mathbf{Z}/(p^{\alpha_i})$ is the canonical map for $1 \leqslant i \leqslant k$. Let $\{e_i\colon\ 1 \leqslant i \leqslant k\}$ be the canonical basis of $(\mathbf{Z}/(p^{\alpha_1}))^k$ and let $z_i = q(e_i)$ for $1 \leqslant i \leqslant k$. Note that the function

$$Y : \left\{x \in \bigwedge^2 G\colon\ i\left(\left(\bigwedge^2 q\right)^{-1}(x)\right) \geqslant k-g\right\} \to \bigcup_{m=k-g}^{[\frac{k}{2}]} U_{k,2m}\big(\mathbf{Z}/(p)\big)$$

defined by

$$Y(x) = \pi X\left(\left(\bigwedge^2 q\right)^{-1}(x)\right)$$

is surjective.

Observe also that the group $\bigoplus_{j=2}^k (p\mathbf{Z}/(p^{\alpha_j}))^{j-1}$ acts freely on $\bigwedge^2 G$ via

$$(x_{ij})_{\substack{1\leqslant i<j \\ 2\leqslant j\leqslant k}} \cdot \sum_{1\leqslant i<j\leqslant k} c_{ij} z_i \wedge z_j = \sum_{1\leqslant i<j\leqslant k} (c_{ij}+x_{ij}) z_i \wedge z_j$$

where $x_{ij} \in p\mathbf{Z}/(p^{\alpha_j})$ for $1 \leqslant i < j \leqslant k$. The function Y defines by passage to the quotient set a bijection

$$\bigoplus_{j=2}^k \big(p\mathbf{Z}/\big(p^{\alpha_j}\big)\big)^{j-1} \Big\backslash \left\{x \in \bigwedge^2 G\colon\ i\left(\left(\bigwedge^2 q\right)^{-1}(x)\right) \geqslant k-g\right\} \to \bigcup_{m=k-g}^{[\frac{k}{2}]} U_{k,2m}\big(\mathbf{Z}/(p)\big)$$

so that

$$\begin{aligned}
\left|\left\{x \in \bigwedge^2 G\colon\ i\left(\left(\bigwedge^2 q\right)^{-1}(x)\right) \geqslant k-g\right\}\right| &= \prod_{j=2}^k \big(p^{\alpha_j - 1}\big)^{j-1}.\sum_{m=k-g}^{[\frac{k}{2}]} \left|U_{k,2m}\big(\mathbf{Z}/(p)\big)\right| \\
&= \bigwedge^2 G.p^{-\binom{k}{2}} \sum_{m=k-g}^{[\frac{k}{2}]} p^{m(2(k-m)-1)}.\frac{\prod_{i=1}^{2m}(1-q^{-(k-2m+i)})}{\prod_{i=1}^{m}(1-q^{-2i})} \\
&= \bigwedge^2 G.\sum_{m\geqslant k-g} p^{-\binom{k-2m}{2}}.\frac{\prod_{i=1}^{2m}(1-q^{-(k-2m+i)})}{\prod_{i=1}^{m}(1-q^{-2i})}
\end{aligned}$$

by virtue of Lemma 3.1. $\square$

Lemma 3.3. *Let α, k be natural numbers and let*

$$C_k(\alpha) = \big\{(\alpha_1, \ldots, \alpha_k) \in \mathbf{Z}^k\colon\ 0 \leqslant \alpha_l \leqslant \cdots \leqslant \alpha_k \leqslant \alpha\big\},$$

then

$$\left|C_k(\alpha)\right| = \binom{\alpha+k}{\alpha}.$$

Proof. Note that the function

$$\theta : C_k(\alpha) \to \left\{ u : [1,k] \to [0,\alpha]\colon \sum_{i=1}^{k} u(i) \leqslant \alpha \right\}$$

defined by $\theta((\alpha_1,\dots,\alpha_k)) = u_{\alpha_1,\dots,\alpha_k}$ where

$$u_{\alpha_1,\dots,\alpha_k} = \begin{Bmatrix} \alpha_1 & i=1 \\ \alpha_i - \alpha_{i-1} & 1 < i \leqslant k \end{Bmatrix}$$

is bijective, therefore our assertion follows from [1, p. 44, Proposition 15]. $\square$

Now we have the following theorem which solves Problem 2 for G a finite Abelian group. It generalizes [7, Theorem 9] and [15, Theorem 2.5].

Theorem 3.4. *Suppose that $G = \bigoplus_{i=1}^n G_i$ is a finite Abelian group and that for, $1 \leqslant i \leqslant n$, G_i is a p_i-Sylow subgroup of G where $p_l < \cdots < p_n$ and let k_i = the minimum number of generators of G_i and $(p_i^{\alpha_i}) = \mathrm{Ann}(G_i)$ and let $q_j : (\mathbf{Z}/(p_i^{\alpha_i})^{k_i} \to G_i$ be any epimorphism, then the bordism map of the free actions of G on the surface $X_{\hat g}$, $\hat g \geqslant 1$, is the composition of the following maps*

$$\begin{aligned} B : \mathrm{Diff}^+(X_{\hat g}) \backslash \mathrm{FA}^+_{\mathrm{Diff}}(G, X_{\hat g}) &\underset{\text{Prop. 0}}{=} H^+(X_{\hat g}) \backslash \mathrm{FA}^+(G, X_{\hat g}) \\ &\underset{\text{Prop. 1.2}}{=} \mathrm{EHom}\bigl(\pi_1(X_g,*), G\bigr) / \mathrm{Aut}^+\bigl(\pi_1(X_g,*)\bigr) \\ &\underset{\text{Prop. 1.3}}{=} \mathrm{EHom}\bigl(\mathbf{Z}^{2g}, G\bigr) / \mathrm{Sp}_{2g}(\mathbf{Z}) \\ &\underset{\text{Prop. 1.6}}{=} \prod_{1\leqslant i\leqslant n} \mathrm{EHom}\bigl(\mathbf{Z}^{2g}, G_i\bigr) / \mathrm{Sp}_{2g}(\mathbf{Z}) \xrightarrow{\Phi} \bigoplus_{1\leqslant i\leqslant n} \bigwedge^2 G_i \\ &\underset{\text{[2, p. 85]}}{=} \bigwedge^2 G \underset{\text{Theorem 1.7}}{=} H_2(G) = \Omega_2(G) \end{aligned}$$

where the map

$$\Phi : \prod_{i=1}^n \mathrm{EHom}\bigl(\mathbf{Z}^{2g}, G_i\bigr) / \mathrm{Sp}_{2g}(\mathbf{Z}) \to \bigoplus_{i=1}^n \bigwedge^2 G_i$$

is defined by

$$\Phi\bigl(([s_i])_{1\leqslant i\leqslant n}\bigr) = \sum_{i=1}^n \bigwedge^2 s_i \left(\sum_{j=1}^g a_j \wedge b_j \right)$$

where $(a_1,\dots,a_g,b_1,\dots,b_g)$ is the canonical ordered basis of $\mathbf{Z}^{2g}$ and g is given by the Riemann–Hurwitz formula $\hat g - 1 = |G|(g-1)$. We have:

(1) *Φ is injective and*

$$\mathrm{Im}\,\Phi = \left\{ \sum_{j=1}^n x_j\colon x_j \in \bigwedge^2 G_j \text{ and } i\left(\left(\bigwedge^2 q_j\right)^{-1} (x_j) \right) \geqslant k_j - g \text{ for } 1 \leqslant j \leqslant n \right\}.$$

(2) $$\bigl|H^+(X_{\hat g}) \backslash \mathrm{FA}^+(G, X_{\hat g})\bigr| = \prod_{j=1}^n \left| \left\{ x \in \bigwedge^2 G_j\colon i\left(\left(\bigwedge^2 q_j\right)^{-1} (x) \right) \geqslant k_j - g \right\} \right|$$

where

$$\left|\left\{x \in \bigwedge^2 G_j \colon i\left(\left(\bigwedge^2 q_j\right)^{-1}(x)\right) \geqslant k_j - g\right\}\right|$$
$$= \left\{\begin{array}{ll} |\bigwedge^2 G_j| & k_j \leqslant g \\ |\bigwedge^2 G_j| . \sum_{m \geqslant k_j - g} p_j^{-\binom{k_j - 2m}{2}} . \frac{\prod_{i=1}^{2m}(1 - p_j^{-(k_j - 2m + i)})}{\prod_{i=1}^{m}(1 - p_j^{-2i})} & k_j > g \end{array}\right\}.$$

Proof. The definition of the bordism map B follows from the computation of Theorem 1.7. The fact that Φ is injective and that $\operatorname{Im}\Phi$ is as described follows from Theorem 2.14. The formula for $|H^+(X_{\hat{g}}) \backslash \mathrm{FA}^+(G, X_{\hat{g}})|$ follows therefore from Lemma 3.2. □

The solution of Problem 3 for the special case of a finite Abelian group whose all p-Sylow subgroups are homocyclic (i.e. of the form $(\mathbf{Z}/p^\alpha\mathbf{Z})^n$) will follow easily from the following more general theorem.

Theorem 3.5. *Let A be a principal ring and let p be an extremal element of A (= a generator of a prime ideal of A). Let V be a free A-module of* rank $2g$ *and let $B = \{a_1, \ldots, a_g, b_l, \ldots, b_g\}$ be a basis of V and let β be a bilinear form on V defined on $B \times B$ by $\beta(a_i, b_i) = -\beta(b_i, a_i) = 1$ for $1 \leqslant i \leqslant g$ and zero otherwise and let $E = \{e_i \colon 1 \leqslant i \leqslant k\}$ be the canonical basis of $(A/(p^\alpha))^k$, then a complete system of representatives of the orbits of* $\mathrm{EHom}(V, (A/(p^\alpha))^k)$ *under the action of* $\mathrm{Aut}_A((A/(p^\alpha))^k \times (\mathrm{Sp}_{2g}(A))^0$ [2, p. 55] *is*:

$$T\big((A/(p^\alpha))^k\big) = \left\{\begin{array}{l} \left\{ f \in \mathrm{EHom}(V, (A/(p^\alpha))^k) \colon\ f(a_j) = \begin{cases} e_j & 1 \leqslant j \leqslant k \\ 0 & \textit{otherwise} \end{cases} \right. \\ \qquad f(b_j) = \begin{cases} p^{\alpha_{\frac{1}{2}(j+1)}} e_{j+1} & 1 \leqslant j \textit{ odd} < k \\ 0 & \textit{otherwise} \end{cases} \\ \qquad \left. 0 \leqslant \alpha_1 \leqslant \cdots \leqslant \alpha_{[k/2]} \leqslant \alpha \right\} \quad \textit{if } k \leqslant g \\ \\ \left\{ f \in \mathrm{EHom}(V, (A/(p^\alpha))^k) \colon\ f(a_j) = e_j \quad 1 \leqslant j \leqslant g \right. \\ \qquad f(b_j) = \begin{cases} e_{g+j} & 1 \leqslant j \leqslant k - g \\ p^{\alpha_{\frac{1}{2}(j-(k-g)+1)}} e_{j+1} & 1 \leqslant j - (k-g) \textit{ odd} < 2g - k \\ 0 & \textit{otherwise} \end{cases} \\ \qquad \left. 0 \leqslant \alpha_1 \leqslant \cdots \leqslant \alpha_{[\frac{2g-k}{2}]} \leqslant \alpha \right\} \quad \textit{if } k > g \end{array}\right.$$

and

$$\left|\mathrm{Aut}_A\big((A/(p^\alpha))^k\big) \backslash \mathrm{EHom}\big(V, (A/(p^\alpha))^k\big)/\mathrm{Sp}_{2g}(A)\right| = \binom{\alpha + [\frac{1}{2}\min(k, 2g-k)]}{\alpha}.$$

Proof. Note that $k =$ the minimum number of generators of $(A/(p^\alpha))^k$ by Lemma 1.4 so that $\mathrm{EHom}(V, (A/(p^\alpha))^k) \neq \emptyset$ iff $k \leqslant 2g$. Observe that $\mathrm{Aut}_A(A/(p^\alpha))^k$ acts on $\bigwedge^2(A/(p^\alpha))^k$ by $\theta.x = \bigwedge^2\theta(x)$ for $\theta \in \mathrm{Aut}_A(A/(p^\alpha))^k$ and $x \in \bigwedge^2(A/(p^\alpha))^k$, so that the bijection

$$s \colon \mathrm{EHom}\big(V, \big(A/\big(p^\alpha\big)\big)^k\big)/\mathrm{Sp}_{2g}(A) \to \left\{x \in \bigwedge^2\big(A/\big(p^\alpha\big)\big)^k \colon i(x) \geqslant k - g\right\}$$

defined by

$$s\big([f]\big) = \sum_{i=1}^{g} f(a_i) \wedge f(b_i)$$

which is given by Corollary 2.9 and Lemma 2.12 induces by passage to quotients a bijection which we also denote by s, by abuse of notation,

$$s : \mathrm{Aut}_A\big(\big(A/\big(p^\alpha\big)\big)^k\big) \backslash \mathrm{EHom}\big(V, \big(A/\big(p^\alpha\big)\big)^k\big) / \mathrm{Sp}_{2g}(A)$$
$$\to \mathrm{Aut}_A\big(\big(A/\big(p^\alpha\big)\big)^k\big) \Big\backslash \Big\{x \in \bigwedge^2 \big(A/\big(p^\alpha\big)\big)^k\colon\ i(x) \geqslant k-g\Big\}.$$

Note that by [4, p. 79, Theorem 1] and since $p^\alpha A$ is a characteristic submodule of A, every orbit of $\mathrm{Aut}_A((A/(p^\alpha))^k)$ on $\{x \in \bigwedge^2 (A/(p^\alpha))^k\colon\ i(x) \geqslant k-g\}$ contains an element of the set

$$T_0 = \Big\{ \sum_{1 \leqslant j \text{ odd} < k} p^{\alpha_{\frac{1}{2}(j+1)}} e_j \wedge e_{j+1}\colon\ 0 \leqslant \alpha_l \leqslant \cdots \leqslant \alpha_{[k/2]} \leqslant \alpha \Big\} \quad \text{if } k \leqslant g$$

$$\text{(respectively} = \Big\{ \sum_{i=1}^{k-g} e_i \wedge e_{g+i} + \sum_{1 \leqslant j-(k-g) \text{ odd} < 2g-k} p^{\alpha_{\frac{1}{2}(j-(k-g)+1)}} e_j \wedge e_{j+1}\colon\ 0 \leqslant \alpha_l \leqslant \cdots \leqslant \alpha_{[(2g-k)/2]} \leqslant \alpha \Big\}$$
$$\text{if } k > g).$$

Suppose that there exists $0 \leqslant \alpha_1^i \leqslant \cdots \leqslant \alpha_{[k/2]}^i \leqslant \alpha$ if $k \leqslant g$ (respectively $0 \leqslant \alpha_1^i \leqslant \cdots \leqslant \alpha_{[(2g-k)/2]}^i \leqslant \alpha$ if $k > g$) for $i = 1, 2$ such that

$$\sum_{1 \leqslant j \text{ odd} < k} p^{\alpha^1_{(j+1)/2}} e_j \wedge e_{j+1} \quad \text{and} \quad \sum_{1 \leqslant j \text{ odd} < k} p^{\alpha^2_{(j+1)/2}} e_j \wedge e_{j+1} \quad \text{if } k \leqslant g$$

$$\text{(respectively } \sum_{i=1}^{k-g} e_i \wedge e_{g+i} + \sum_{1 \leqslant j-(k-g) \text{ odd} < 2g-k} p^{\alpha^1_{\frac{1}{2}(j-(k-g)+1)}} e_j \wedge e_{j+1} \quad \text{and}$$
$$\sum_{i=1}^{k-g} e_i \wedge e_{g+i} + \sum_{1 \leqslant j-(k-g) \text{ odd} < 2g-k} p^{\alpha^2_{\frac{1}{2}(j-(k-g)+1)}} e_j \wedge e_{j+1} \quad \text{if } k > g)$$

are in the same orbit under $\mathrm{Aut}_A((A/(p^\alpha))^k)$, then if we let the bilinear forms φ^i, $i = 1, 2$, on $(A/(p^\alpha))^k$ be defined on $E \times E$ by $\varphi^i(e_j, e_{j+1}) = -\varphi^i(e_{j+1}, e_j) = p^{\alpha^i_{\frac{1}{2}(j+1)}}$ for $1 \leqslant j$ odd $< k$ and is zero otherwise for $k \leqslant g$ (respectively $\varphi^i(e_j, e_{g+j}) = -\varphi^i(e_{g+j}, e_j) = 1$ for $1 \leqslant j \leqslant k-g$ and $\varphi^i(e_j, e_{j+1}) = -\varphi^i(e_{j+1}, e_j) = p^{\alpha i(j-(k-g)+1)/2}$ for $1 \leqslant j-(k-g)$ odd $< 2g-k$ and is zero otherwise for $k > g$) we would have $d_{\varphi^2} = {}^t\theta d_{\varphi^1}\theta$ for some $\theta \in \mathrm{Aut}_A((A/(p^\alpha))^k)$, where $d_{\varphi^2} : (A/(p^\alpha))^k \to ((A/(p^\alpha))^k)^*$, $i = 1, 2$, are the partial maps defined by $d_{\varphi^i}(x) = \varphi^i(\cdot, x)$. It follows that, if we set $R = A/(p^\alpha)$, then

$$\big(R^k\big)^*/d_{\varphi^1}\big(R^k\big) \cong \big(R^k\big)^*/d_{\varphi^2}\big(R^k\big)$$

and since

$$\big(R^k\big)^*/d_{\varphi^i}\big(R^k\big) \cong \begin{Bmatrix} \bigoplus_{j=1}^{[k/2]} (R/p^{\alpha^i_j}R)^2 \oplus R & \text{if } k \text{ odd} \\ \bigoplus_{j=1}^{[k/2]} (R/p^{\alpha^i_j}R)^2 & \text{if } k \text{ even} \end{Bmatrix}$$
$$\cong \begin{Bmatrix} \bigoplus_{j=1}^{[k/2]} (A/p^{\alpha^i_j})^2 \oplus A/(p^\alpha) & \text{if } k \text{ odd} \\ \bigoplus_{j=1}^{[k/2]} (A/p^{\alpha^i_j})^2 & \text{if } k \text{ even} \end{Bmatrix} \quad \text{if } k \leqslant g$$

$$\text{(respectively } \big(R^k\big)^*/d_{\varphi^i}\big(R^k\big) \cong \begin{Bmatrix} \bigoplus_{j=1}^{[\frac{1}{2}(2g-k)]} (R/p^{\alpha^i_j}R)^2 \oplus R & \text{if } 2g-k \text{ odd} \\ \bigoplus_{j=1}^{[\frac{1}{2}(2g-k)]} (R/p^{\alpha^i_j}R)^2 & \text{if } 2g-k \text{ even} \end{Bmatrix}$$
$$\cong \begin{Bmatrix} \bigoplus_{j=1}^{[\frac{1}{2}(2g-k)]} (A/p^{\alpha^i_j})^2 \oplus A/(p^\alpha) & \text{if } 2g-k \text{ odd} \\ \bigoplus_{j=1}^{[\frac{1}{2}(2g-k)]} (A/p^{\alpha^i_j})^2 & \text{if } 2g-k \text{ even} \end{Bmatrix} \quad \text{if } k > g)$$

for $i = 1, 2$, [3, p. 16, Corollary 1] shows that we must have $\alpha_j^1 = \alpha_j^2$ for $1 \leqslant j \leqslant [k/2]$ if $k \leqslant g$ (respectively $\alpha_j^1 = \alpha_j^2$ for $1 \leqslant j \leqslant [(2g-k)/2]$ if $k > g$).

The assertion of the theorem follows since the set $T((A/(p^\alpha))^k)$ maps under s bijectively onto the set T_0. The formula for $|\operatorname{Aut}_A((A/(p^\alpha))^k)\backslash \operatorname{EHom}(V, (A/(p^\alpha))^k)/\operatorname{Sp}_{2g}(A)|$ follows therefore by virtue of Lemma 3.3. □

The solution of Problem 3 of our classification problem for the special case mentioned earlier can now be given in the following theorem. It generalizes [17, p. 502] and [7, Corollary 10] and corrects [15, Proposition 4.6] and [10, Remark 4.5].

Theorem 3.6. *Let $G = \bigoplus_{i=1}^n (\mathbf{Z}/(p_i^{\alpha_i}))^{k_i}$ where $k_i \geqslant 1$, $1 \leqslant i \leqslant n$ and $p_1 < \cdots < p_n$ and $\{p_1, \ldots, p_n\}$ is the set of prime divisors of $|G|$ and let $\hat{g} \geqslant 1$, then we have*

$$
\begin{aligned}
&\operatorname{Diff}^+(X_{\hat{g}})\backslash \operatorname{FA}^+_{\mathrm{Diff}}(G, X_{\hat{g}})/\operatorname{Aut}(G)\\
&\quad \underset{\text{Prop. 0}}{=} H^+(X_{\hat{g}})\backslash \operatorname{FA}^+(G, X_{\hat{g}})/\operatorname{Aut}(G)\\
&\quad \underset{\text{Prop. 1.2}}{=} \operatorname{Aut}(G)\backslash \operatorname{EHom}\big(\pi_1(X_g, *), G\big)/\operatorname{Aut}^+\big(\pi_1(X_g, *)\big)\\
&\quad \underset{\text{Prop. 1.3}}{=} \operatorname{Aut}(G)\backslash \operatorname{EHom}\big(\mathbf{Z}^{2g}, G\big)/\operatorname{Sp}_{2g}(\mathbf{Z})\\
&\quad \underset{\text{Prop. 1.6}}{=} \prod_{i=1}^n \operatorname{Aut}\big(\big(\mathbf{Z}/\big(p_i^{\alpha_i}\big)\big)^{k_i}\big)\backslash \operatorname{EHom}\big(\mathbf{Z}^{2g}, \big(\mathbf{Z}/\big(p_i^{\alpha_i}\big)\big)^{k_i}\big)/\operatorname{Sp}_{2g}(\mathbf{Z})
\end{aligned}
$$

and for $1 \leqslant i \leqslant n$, the set $T((\mathbf{Z}/(p_i^{\alpha_i}))^{k_i})$ defined in Theorem 3.5 is a set of complete system of representatives of the orbits of $\operatorname{Aut}((\mathbf{Z}/(p_i^{\alpha_i}))^{k_i}) \times (\operatorname{Sp}_{2g}(\mathbf{Z}))^0$ [2, p. 55] on $\operatorname{EHom}(\mathbf{Z}^{2g}, (\mathbf{Z}/(p_i^{\alpha_i}))^{k_i})$, hence

$$
\big|H^+(X_{\hat{g}})\backslash \operatorname{FA}^+(G, X_{\hat{g}})/\operatorname{Aut}(G)\big| = \prod_{i=1}^n \binom{\alpha_i + [\frac{1}{2}\min(k_i, 2g-k_i)]}{\alpha_i}
$$

where g is given by the Riemann–Hurwitz formula $\hat{g} - 1 = |G|(g-1)$.

Proof. All our assertions follow directly from Theorem 3.5. □

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